ON THE COMASS OF FORMS-PRODUCTS

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1. INTRODUCTION

The calibration method was studied systematically by Dao Trong Thi in [3, 4] and R. Harvey, B. Lawson in [7]. Various concrete calibrations were used by many authors to find minimal faces, see Federer [5] Berger [1], Dao Trong Thi [3, 4], Harvey - Lawson [7], Dadok - Harve- organ [2], Le Hong Van [9] Hoang Xuan Huan [8] etc... For applying this method, determining comase and the maximal directions of a p-covector is the main obstacle. In the field mentioned ove, it is still open the question whether the equality $\|\varphi \wedge \psi\|^* = \|\varphi\|^*.\|\psi\|^*$ holds where φ and are forms on orthogonal subspaces of R^n. Let $\varphi \in \Lambda^k R^m$, $\psi \in \Lambda^l R^n$, then $\varphi \wedge \psi \in \Lambda^{k+l} R^{m+n}$. note that although the inequality $\|\varphi \wedge \psi\|^* \geq \|\varphi\|^*.\|\psi\|^*$ is obvious the equality $\|\varphi \wedge \psi\|^* =$ $\|^*.\|\psi\|^*$ had been proved only for some concrete cases. Morgan [11] has showed that the equality lds if $k \leq 2$, or $m-k \leq 2$, or $k = l = 3$, or $m - k = n - l = 3$. Recently, Hoang Xuan Huan [8] s proved it for an arbitrary E-separable form φ. In this paper we prove the equality when φ is her a torus form, or a certain averaged form by a group.

2. THE COMASS OF A PRODUCT WITH A FACTOR BEING A TORUS FORM

First we recall some notions and facts of exterior algebra.

Let R^n be the n-dimensional Euclidean space, $\Lambda_k R^n$ and $\Lambda^k R^n$ the dual spaces of the k-ctors and the k-covectors respectively. The inner product and the norm on R^n induce the inner oduct and the norm on $\Lambda_k R^n$ and $\Lambda^k R^n$. Consider an orthonormal basis $e_1, \dots e_n$ of R^n and dual basis $e_1^*, \dots, e_n^*$ of $\Lambda^1 R^n$ (from now on, the symbol e^* means the dual covector of e), en an arbitrary p-covector in R^n has an unique expression $\varphi = \sum a_I e_I^*$, where $I = (i_1, \dots, i_p)$, $\leq i_1 < \cdots i_p \leq n$ and $e_I^* = e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*$. The comass of a p-covector φ is defined by

$$\|\varphi\|^* = \sup\{\varphi(\xi) : \xi \in G(p, R^n)\},$$

ere the Grassmannian $G(p, R^n)$ consists of all oriented p-planes in R^n and it may be identified th the collection of unit simple p-vectors in R^n.

For any p-covector φ in R^n the set of maximal directions of φ is defined by

$$G(\varphi) = \{\xi \in G(p, R^n) : \varphi(\xi) = \|\varphi\|^*\}.$$

Let φ be a p-covector in a subspace $V \subset R^n$ then φ can be considered as a p-covector in R^n identifying with $\Pi^* \varphi$ where Π is the orthogonal projection of R^n on V.

Because every p-covector in R^n can be considered as a parallel differential p-form in R^n, from w on, we shall call every p-covector in R^n to be a p-form in R^n unless otherwise stated.

Now we recall the notion of torus form which was introduced and its comass was com[
in several papers, for example, see [2]. Here we consider it only in the relation with the pr[
mentioned above.

Definition 1. Identify $R^{2n} \cong C^n$ with real orthonormal basis $e_1, Je_1, \ldots, e_n, Je_n$. Any n-fo[
is called a torus form on R^{2n} if it belongs to $\bigotimes_{k=1}^{n} \Lambda^1 \operatorname{span}(e_k, Je_k) \subset \Lambda^n R^{2n}$.

Note that the Lagrangian forms $\operatorname{Re} e^{i\theta} dz_1 \wedge \cdots \wedge dz_n$ are torus forms.

The torus forms belong to the class of V-torus forms defined below:

Definition 2. Let V be a 2-dimensional subspace of R^n $(n \geq 2)$. Any form $\varphi \in \Lambda^1 V \oplus$ [
$(\ell \geq 0)$ is called a V-torus form.

We note that if e_1, e_2 is an orthonormal basis of V then each V-torus form can be expr[
as

$$\varphi = e_1^* \wedge \varphi_1 + e_2^* \wedge \varphi_2 ,$$

where φ_1 and φ_2 are forms on $V^\perp$.

Obviously, for any p-form Ω on R^n, $\|\Omega\|^* = \max_{v \in R^n, |v|=1} \|v \lrcorner \Omega\|^*$. Moreover, for V-torus [
we have the following

Lemma 1. *Let Ω be a V-torus form on R^n. Then* $\|\Omega\|^* = \max_{v \in V, |v|=1} \|v \lrcorner \Omega\|^*$.

Proof. The inequality $\max_{v \in V, |v|=1} \|v \lrcorner \Omega\|^* \leq \|\Omega\|^*$ is evident.

Conversely, take $\xi \in G(\Omega)$ and put ξ in canonical form with respect to V (see [7, Le[
7.5]), that is

$$\xi = (\cos\theta_1\, e_1 + \sin\theta_1\, f_1) \wedge (\cos\theta_2\, e_2 + \sin\theta_2\, f_2) \wedge f_3 \wedge \cdots \wedge f_p ,$$

where e_1, e_2 is and orthonormal basis of V, $f_1, f_2, \ldots, f_p$ are orthonormal vectors in $V^\perp$ and a[
θ_i, $0 \leq \theta_i \leq \frac{\pi}{2}$, for $i = 1, 2$. Then

$$\begin{aligned}
\Omega(\xi) &= \cos\theta_1\, \sin\theta_2\, \Omega\,(e_1 \wedge f_2 \wedge \cdots \wedge f_p) + \sin\theta_1\, \cos\theta_2\, \Omega\,(f_1 \wedge e_2 \wedge \cdots \wedge f_p) \\
&= a\, \cos\theta_1\, \sin\theta_2 + b\, \sin\theta_1\, \cos\theta_2 \leq \max\{|a|, |b|\}\, (\cos\theta_1\, \sin\theta_2 + \sin\theta_1\, \cos\theta_2) \\
&= \max\{|a|, |b|\}\, \sin(\theta_1 + \theta_2) \leq \max\{|a|, |b|\} \leq \|\Omega\|^* ,
\end{aligned}$$

where $a = \Omega\,(e_1 \wedge f_2 \wedge \cdots \wedge f_p)$, $b = \Omega\,(f_1 \wedge e_2 \wedge \cdots \wedge f_p)$.

Hence the inequalities become equality, in particular, $\max\{|a|, |b|\} = \|\Omega\|^*$.

Therefore, $|a| = \|\Omega\|^*$ or $|b| = \|\Omega\|^*$.

But $|a| = |\Omega\,(e_1 \wedge f_2 \wedge \cdots \wedge f_p)| = |e_1 \lrcorner \Omega\,(f_2 \wedge \cdots \wedge f_p)| \leq \|e_1 \lrcorner \Omega\|^*$

and $|b| = |\Omega\,(f_1 \wedge e_2 \wedge \cdots \wedge f_p)| = |-e_2 \lrcorner \Omega\,(f_1 \wedge \cdots \wedge f_p)| \leq \|-e_2 \lrcorner \Omega\|^*$,

therefore $\|\Omega\|^* \leq \|e_1 \lrcorner \Omega\|^*$ or $\|\Omega\|^* \leq \|-e_2 \lrcorner \Omega\|^*$.

Thus, we have $\|\Omega\|^* = \max_{v \in V, |v|=1} \|v \lrcorner \Omega\|^*$.

The lemma is proved.

Lemma 2. *Let $R^{n+m} = R^n \oplus R^m$ be an orthogonal decomposition of R^{n+m}. Let $\varphi \in \Lambda$[
$\psi \in \Lambda^q R^m$, and $v \in R^{n+m}$ such that $v \lrcorner \psi = 0$. Then $v \lrcorner (\varphi \wedge \psi) = (v \lrcorner \varphi) \wedge \psi$.*

Suppose $v_2, ..., v_{p+q} \in R^{n+m}$ and $v_1 = v$. We have

$$v \rfloor (\varphi \wedge \psi)(v_2, ..., v_{p+q}) = (\varphi \wedge \psi)(v_1, v_2, ..., v_{p+q})$$
$$= \sum_{\sigma \in Sh(p,q)} \text{index}(\sigma)\, \varphi(v_{\sigma(1)}, ..., v_{\sigma(p)})\, \psi(v_{\sigma(p+1)}, ..., v_{\sigma(p+q)}),$$

$Sh(p, q)$ consists of all permutations σ of $\{1, ..., p+q)\}$ such that σ increases on the set of $p\}$ and the set of $\{p+1, ..., p+q\}$.

ince $v_1 \rfloor \psi = 0$ the above sum equals to

$$\sum_{\sigma \in Sh(p-1,q)} \text{index}(\sigma)\, \varphi(v_1, v_{\sigma(2)}, ..., v_{\sigma(p)})\, \psi(v_{\sigma(p+1)}, ..., v_{\sigma(p+q)})$$
$$= \sum_{\sigma \in Sh(p-1,q)} \text{index}(\sigma)(v_1 \rfloor \varphi)(v_{\sigma(2)}, ..., v_{\sigma(p)})\, \psi(v_{\sigma(p+1)}, ..., v_{\sigma(p+q)})$$
$$= (v_1 \rfloor \varphi) \wedge \psi(v_2, ..., v_p, ..., v_{(p+q)}),$$

e σ is a permutation of $\{2, 3, ..., p+q\}$.

Consequently $v_1 \rfloor (\varphi \wedge \psi) = (v_1 \rfloor \varphi) \wedge \psi$. The lemma is proved.

orem 1. *Let be φ a torus form on $R^{2n} \cong C^n$ and ψ be a p-form on R^m. Consider $\Omega = \varphi \wedge \psi$ a $(n+p)$-form on $R^{2n} \oplus R^m$. Then $\|\Omega\|^* = \|\varphi\|^*.\|\psi\|^*$ and $G(\Omega) \supset G(\varphi) \wedge G(\psi)$.*

ark. The second conclusion holds for every case when the first one holds.

. Let $e_1, Je_1, ..., e_n, Je_n$ is an orthonormal basis of R^{2n} and $\varphi \in \bigotimes_{k=1}^{n} \Lambda^1 \text{span}(e_k, Je_k) \subset$ ^{n}. Then φ and $\varphi \wedge \psi$ are V-torus forms, where $V = \text{span}(e_1, Je_1)$. We note that $v \rfloor \varphi$ where is a torus form on $R^{2(n-1)} = V^{\perp}$. Using Lemma 1, Lemma 2 by induction on n we have:

$$\|\varphi \wedge \psi\|^* = \max_{v \in V, |v|=1} \|v \rfloor (\varphi \wedge \psi)\|^* = \max_{v \in V, |v|=1} \|(v \rfloor \varphi) \wedge \psi\|^*$$
$$= \max_{v \in V, |v|=1} \|v \rfloor \varphi\|^* \, \|\psi\|^* = \|\varphi\|^* \, \|\psi\|^*.$$

take $\eta \in G(\varphi)$ and $\lambda \in G(\psi)$, we have

$$\Omega\,(\eta \wedge \lambda) = \varphi(\eta).\psi(\lambda) = \|\varphi\|^* \, \|\psi\|^* = \|\Omega\|^*$$

$\eta \wedge \lambda \in G(\Omega)$, this implies that $G(\Omega) \supset G(\varphi) \wedge G(\psi)$.

Every parallel differential form having comass one is a calibration (the notion of calibration e given later in section 4). The following corollary follows directly from Theorem 1.

llary 1. *Let $\text{Re}dz = \text{Re}dz_1 \wedge \cdots \wedge dz_n$ be the special Lagrangian calibration on $R^{2n} \cong C^n$ 7]) and φ be a calibration on R^m. Then $\omega = \text{Re}dz \wedge \varphi$ is a calibration on R^{2n+m} and $\supset S(\text{Lag}) \wedge G(\varphi)$ where $S(\text{Lag})$ consists of all special Lagrangian subspaces of $R^{2n} \cong C$.*

Because for an arbitrary 3-form on R^6 there is a convenient basis so that this 3-form is a torus (see [11] , from Theorem 5.1 in [11] and Theorem 1 we have the following

Corollary 2. *Let φ be a calibration on R^6 and ψ be a calibration on R^m. Then φ calibration on R^{6+m} and $G(\varphi \wedge \psi) \supset G(\varphi) \wedge G(\psi)$.*

3. THE COMASS OF A PRODUCT WITH A FACTOR BEING AN AVERAGED F

Let $\mathcal{G} \subset O(n)$ be a compact Lie group, each k-form $\varphi = \int_{\mathcal{G}} g^* \omega \, dg$ for any $\omega \in \Lambda^k R^n$ an averaged form by group $\mathcal{G}$. Some known averaged forms are the normalized powers o nionics Kahler forms, the Euler forms and their "adjusted powers". Using them as cal one showed certain submanifolds are homologically minimal in quaternionic Kahler manif in Grassmannian manifold (see [6, 12].

In this section we prove that the equality on the comass of a product holds when o is a certain averaged form.

Let $\mathcal{G}$ be a compact Lie group. Consider the Haar measure on $\mathcal{G}$ such that the measu whole group $\mathcal{G}$ equals to 1. We have the following

Theorem 2. *Let $\mathcal{G} \subset O(n)$ be a compact Lie subgroup and $\omega \in \Lambda^k R^n$, suppose that $\xi \in$ (spanξ is $\mathcal{G}$-invariant. Then*

$$\left\| \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega \, dg \right\|^* = \|\omega\|^*$$

and ξ is a maximal direction of the form on the left side.

Proof. Note that since spanξ is $\mathcal{G}$-invariant and $\mathcal{G} \subset O(n)$, $\det(g|_{\mathrm{span}\xi}) = +1$ or -1 we h

$$\left\| \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega \, dg \right\|^* \leq \|\omega\|^* .$$

Indeed, take $\eta \in G(k, R^n)$, then

$$\left| \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega(\eta) \, dg \right| \leq \int_{\mathcal{G}} |\omega(g_* \eta)| \, dg \leq \int_{\mathcal{G}} \|\omega\|^* \, dg = \|\omega\|^*$$

for any $\eta \in G(k, R^n)$, therefore

$$\left\| \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega \, dg \right\|^* \leq \|\omega\|^* .$$

Conversely, we have $g_* \xi = \det(g|_{\mathrm{span}\xi}).\xi$.
Therefore

$$\left\| \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega \, dg \right\|^* \geq \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega(\xi) \, dg = \int_{\mathcal{G}} \det{}^2(g|_{\mathrm{span}\xi})\, \omega(\xi) \, dg .$$

Since $\xi \in G(\omega)$ and $\det(g|_{\mathrm{span}\xi}) = 1$ or -1 we have

$$\left\| \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega \, dg \right\|^* \geq \int_{\mathcal{G}} \|\omega\|^* \, dg = \|\omega\|^* .$$

Hence,

$$\left\| \int_{\mathcal{G}} \det(g|_{\mathrm{span}\xi})\, g^* \omega \, dg \right\|^* = \|\omega\|^*$$

is a maximal direction of the form on the left side. The theorem is proved.

em 3. *Let* $\varphi \in \Lambda^k R^n$, $\psi \in \Lambda^l R^m$, $\varphi \wedge \psi \in \Lambda^{k+l} R^{n+m}$ *such that* $\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*$. $\subset O(n)$ *be a compact Lie subgroup such that* $\| \int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg\|^* = \|\varphi\|^*$, *where* $\chi(g)$ *is a* *on of value 1 or -1 on* $\mathcal{G}$. *Then*

$$\|(\int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg) \wedge \psi\|^* = \| \int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg\|^* . \|\psi\|^* .$$

It is sufficient to prove the inequality

$$\|(\int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg) \wedge \psi\|^* \leq \| \int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg\|^* . \|\psi\|^* .$$

ve

$$\|(\int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg) \wedge \psi\|^* = \| \int_{\mathcal{G}} (\chi(g)\, g^* \varphi \wedge \psi)\, dg\|^* = \| \int_{\mathcal{G}} \chi(g)\, g^* (\varphi \wedge \psi)\, dg\|^* , \quad (*)$$

here $\mathcal{G}$ is considered as a subgroup of $O(n+m)$ such that $g|_{R^m} = i\, d_{R^m}$, for any $g \in \mathcal{G}$. rom $(*)$ it follows that

$$\begin{aligned} \|(\int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg) \wedge \psi\|^* &\leq \int_{\mathcal{G}} \|(\chi(g)\, g^* \varphi \wedge \psi)\|^*\, dg \\ &= \int_{\mathcal{G}} \|\varphi \wedge \psi\|^*\, dg = \|\varphi \wedge \psi\|^* . \quad (**) \end{aligned}$$

$\|\varphi \wedge \psi\|^* = \|\varphi\|^* . \|\psi\|^*$ from $(**)$ we have

$$\|(\int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg) \wedge \psi\|^* \leq \|\varphi\|^* . \|\psi\|^* = \| \int_{\mathcal{G}} \chi(g)\, g^* \varphi\, dg\|^* . \|\psi\|^* .$$

ieorem is proved.

ow we apply the above results for the powers of quaternionic Kahler forms, the Euler forms ieir "adjusted powers".

ne showed that submanifolds $G_k R^{k+p}$ is homologically minimal in Grassmannian manifold $^{+n}$ for k-even and they are calibrated by forms λ_p given below (see [6]).

et us consider an orthogonal complex structure J on R^k then J defines a complex structure same name on $R^k \otimes R^n$ by $J(u \otimes v) = J(u) \otimes v$. Let ω_J denote the corresponding Kahler form $\otimes R^n$. Consider kp-form $\Omega = \frac{\omega_J^{rp}}{(rp)!}$ $(k = 2r)$. A twisted average of these powers of Kahler by the space of all possible complex structure J on R^k, equivalently, by the group $O(k)$ as s. Fix the complex structure J on R^k. Then for each $g \in O(k)$ consider the corresponding ex structure $g^{-1} J g$ on R^k and by our convention, also on $R^k \otimes R^n$, we have $g^* \omega_J = \omega_{g^{-1} J g}$. ler the form

$$\lambda_p = \int_{O(k)} (\det)^p\, g^*\, \Omega\, dg .$$

λ_p is a $SO(k) \times SO(n)$-invariant form, hence it induces a $SO(k+n)$-invariant differential on $G_k R^{k+n}$. When $p = 1$ it is called the Euler form and when $p > 1$ it is an "adjusted power" of the Euler form (the term of "adjusted power" was used by Le Hong Van in [10], b the above construction, see [6].

Let $e_1, \ldots, e_k$ is an oriented orthonormal basis in R^k such that this orientation agrees canonical orientation of complex structure J. Let $f_1, \ldots, f_n$ is an orthonormal basis of R^n. '

$$\xi = (e_1 \otimes f_1) \wedge \cdots \wedge (e_k \otimes f_1) \wedge \cdots \wedge (e_1 \otimes f_p) \wedge \cdots \wedge (e_k \otimes f_p), \text{ for } p \leq n$$

is a canonically oriented complex rp-plane for complex structure J in $R^k \otimes R^n$, i.e. $\xi \in$ and spanξ is $O(k)$-invariant. Each $g \in O(k)$ is extended on $R^k \otimes R^n$ by $g(u \otimes v) = g(u) \otimes v$ $\det(g|_{\text{span}\xi} = (\det g)^p$ (on the right side, we consider the determinant of transformation g or Applying Theorem 2, Theorem 3 and Theorem 6.1 in [8] for powers of the Kahler forms we c the following.

Corollary 3. *Let $\lambda_p = \int_{O(k)} (\det g)^p\, g^* \Omega\, dg$ be the form mentioned above. Then*

$$\|\lambda_p\|^* = \|\Omega\|^* = 1$$

and

$$\|\lambda_p \wedge \psi\|^* = \|\lambda_p\|^* . \|\psi\|^*,$$

where ψ is any form on a space orthogonal to $R^k \otimes R^n$.

In fact, the first conclusion of Corollary 3 had been proved in [6].

Now we consider the quaternionic Kahler form on a quaternionic Kahler manifold it is d as follows: on a quaternionic Kahler manifold M and parallel differential form determin $Sp(n) \times Sp(1)$-invariant form $\Omega = \frac{1}{6}(\Omega_I^2 + \Omega_J^2 + \Omega_K^2)$ on H^n is called the quaternionic K form and it is also denoted by Ω, where ω_I, ω_J, ω_K are Kahler form corresponding to co structures I, J, K on H^n and where H^n is identified with a tangent space $T_x M$ for some x (see [12]).

Tasaki [12] has proved that

$$\frac{\Omega^m}{m!} = \int_{z \in Sp(1)} z^* \frac{\Omega_I^{2m}}{(2m)!} dz$$

($Sp(1) \equiv \{z \in H, |z| = 1\}$ and $Sp(1)$ acts on H^n by the left handed multiplication).

We note that $\xi = v_1 \wedge v_1 i \wedge v_1 j \wedge v_1 k \wedge \cdots \wedge v_m \wedge v_m i \wedge v_m j \wedge v_m k$, where $v_1, \ldots, v$ system of H-linearly independent orthonormal vectors in H^n, satisfies the conditions in Th 2 for $\omega = \dfrac{\Omega_J^{2m}}{(2m)!}$ and $\mathcal{G} = Sp(1)$. Further since $Sp(1)$ acts on span v_p, $v_p i$, $v_p j$, $v_p k)$ for $p = 1, 2, \ldots, m$ with the determinants of transformations equal to 1, therefore $\det(g|_{\text{span}\xi}) =$ any $g \in Sp(1)$. Because of the same reason as for corollary 3 we obtain the following

Corollary 4. *Let Ω be the form on H^n defined as above. Then*

$$\left\| \frac{\Omega^m}{m!} \right\|^* = \left\| \frac{\Omega_I^{2m}}{(2m)!} \right\|^* = 1$$

and

$$\|\Omega^m \wedge \psi\|^* = \|\Omega^m\|^* . \|\psi\|^*$$

where ψ is any form on a space orthogonal to H^n.

In fact, the first conclusion of Corollary 4 had been proved in [12].

We consider the product $M \times N$ of Riemannian manifolds M and N. Let S and T be minimal nts on M and N respectively. In general, one does not know whether the Cartesian product ' is also minimal on $M \times N$. Applying the calibration method with using results on the comass oducts, we can give some new examples of minimal currents as Cartesian products of minimal nt in the class of normal currents.

First we recall some necessary notions and facts (for details see [4]).

Let φ and S be a differential p-form and a p-current in a Riemannian manifold M respectively. comass of φ is defined by $\|\varphi\|^* = \sup\{\|\varphi_x\|^*,\ x \in M\}$ and the set of maximal directions of φ fined by

$$G(\varphi) = \bigcup \{G(\varphi_x),\ \|\varphi_x\|^* = \|\varphi\|^*\}.$$

If φ is closed and has comass one then it is called a calibration. The mass of S is defined by

$$M(S) = \sup\{S(\varphi),\ \|\varphi\|^* = 1\}$$

is a surface in M then $M(S) = \text{volume}(S)$. A current S in a Riemannian manifold is called ologically minimal with respect to the mass if $M(S) \leq M(S')$ for any current S' homologous , and the S is called homologically mass-minimizing current, or simply, homologically minimal ent. A fundamental theorem of the calibration method [4, Theorem 3.6] says that a current S Riemannian manifold M is homologically minimal if and only if there exists a closed form Ω that the tangent $\vec{S}_x$ of S belongs to $G(\Omega)$ almost every where (in the sence of the measure |. In this case, we say that S is calibrated by Ω.

We have the following

orem 4. *Let S and T be two homologically minimal currents in Riemannian manifolds M and spectively. If S is calibrated by φ and T is calibrated by ψ such that $\|\varphi_x \wedge \psi_y\|^* = \|\varphi_x\|^*\ \|\psi_y\|^*$ iny $x \in M$, $y \in N$. Then $S \times T$ is homologically minimal in $M \times N$.*

iark. φ and ψ can be considered as differential forms on $M \times N$ by identifying with $\pi_1^* \varphi$ and respectively where $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ are canonical projections.

of. We have $\varphi \wedge \psi$ be closed and

$$\begin{aligned}
\|\varphi \wedge \psi\|^* &= \sup\{\|(\varphi \wedge \psi)_{(x,y)}\|^*,\ (x, y) \in M \times N\} \\
&= \sup\{\|(\varphi_x \wedge \psi_y)\|^*,\ x \in M,\ y \in N\} \\
&= \sup\{\|\varphi_x\|^*\ \|\psi_y\|^*,\ x \in M,\ y \in N\} \\
&= \sup_{x \in M} \|\varphi_x\|^* . \sup_{y \in N} \|\psi_y\|^* \\
&= \|\varphi\|^* . \|\psi\|^* = 1.
\end{aligned}$$

refore $\varphi \wedge \psi$ is calibration on $M \times N$.

On the other hand, since $\overrightarrow{S \times T}_{(x,y)} = \vec{S}_x \wedge \vec{T}_y$ and $\varphi(\vec{S}_x) = 1$, $\psi(\vec{T}_y) = 1$ almost every where iave

$$(\varphi \times \psi)(\overrightarrow{S \times T}_{(x,y)}) = \varphi(\vec{S}_x).\psi(\vec{T}_y) = 1 \text{ almost every where.}$$

s shown that $S \times T$ is calibrated by $\varphi \wedge \psi$.

Examples

Example 1. Let S be a special Lagrangian submanifold of R^{2n} and T be a homologically mini current in a Riemannian manifold N. Then $S \times T$ is homologically minimal current in $R^{2n} \times$ This follows form Corollary (3) and Theorem 4.

Example 2. Let S be a homologically minimal current in a 6-dimensional Riemannian manifold and T a homologically minimal current and a Riemannian manifold N. Then from Corollary and Theorem 4 it follows that $S \times T$ is homologically minimal in $M \times N$.

Example 3. Let M be a $4n$-dimensional quaternionic Kahler manifold and S be a quaternic Kahler submanifold of M. Let T be a homologically minimal current in a Riemannian manil N. Since S calibrated by differential form $\frac{\Omega^m}{m!}$ $(m < n)$ where Ω is quaternionic Kahler form ([12]) from Corollary 4 and Theorem 4 it following that $S \times T$ is homologically minimal curren $M \times N$.

Example 4. For k even integer, by Proposition 3.1 in [6] the submanifold $G_k R^{k+p}$ of $G_k R^{l}$ is homologically minimal and it calibrated by form λ_p mentioned in section 3. Let S be a hoi logically minimal current in a Riemannian manifold N, then from Corollary 3 and Theorem follows that $G_k R^{k+p} \times S$ is homologically minimal current in $G_k R^{k+n} \times N$.

Remark. The above results hold, in particular, when currents are replaced by surfaces.

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REFERENCES

1. M. Berger. Du Côté de chez Pu, Ann., Scient. Ec. Norm. Sup., **4** (1972), 1-44.
2. J. Dadok, R. Harvey, and F. Morgan. Calibration on R^8, Trans. Amer. Math. Soc., : (1988), 1-40.
3. Dao Trong Thi. Minimal current on compact manifolds, Izv. Akad. Nauk USSR, Ser. Ma **41** (1977), 853-867 (in Russian).
4. Dao Trong Thi. Globally minimal currents and surfaces in Riemannian manifolds, Acta M: Vietnam., **10** (1985), 296-333.
5. H. Federer. Geometric Measure Theory, Berlin Springer, 1969.
6. H. Gluck, F. Morgan, and W. Ziller. Calibrated geometry in Grassmannian manifolds, Cor Math. Helv., **64** (1989), 256-268.
7. R. Harvey and H. B. Lawson. Calibrated geometries, Acta Math., **148** (1982), 47-157.
8. Hoang Xuan Huan. Separable calibrations and minimal surfaces, Acta Math. Vietnam., (1994), 77-96.
9. Le Hong Van. Minimal surfaces on homogeneous spaces, Izv. Akad. Nauk USSR, Ser. Ma **52** (1988), 1-39.
10. Le Hong Van. Application of integral geometry to minimal surfaces, International Journa Mathematics, Vol. 4, No. 1 (1993), 89-111.

Morgan. The exterior algebra $\Lambda^k R^n$ and area minimization, **Linear Algebra Appl., 66** 35), 1-28.

Tasaki. Certain minimal or homologically volume minimizing submanifolds in compact metric space, Tsukuba J. Math., Vol. 9, No. 1, (1985), 117-131.

KHOA HỌC ĐHQGHN, KHTN, t.XII, n°1, 1996

VỀ ĐỐI KHỐI LƯỢNG CỦA TÍCH CÁC DẠNG

Nguyễn Duy Bình
Khoa Toán - Đại học Sư phạm Vinh

eo phương pháp dạng cỡ, vấn đề xác định khối lượng và các hướng cực đại của tích các ên các không gian trực giao có ý nghĩa quan trọng trong việc tìm các mặt cực tiểu thể o hàm tích các mặt cực tiểu thể tích. Trong bài này chúng tôi chứng minh một đẳng thức lượng của tích các dạng khi một nhân tử là một dạng xuyến hoặc là một dạng trung bình bởi một nhóm. Áp dụng kết quả này, chúng tôi nhận thấy được một số ví dụ về các mặt u thể tích.