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REMARKS ON THE NEWTON - KANTOROVIC'S METHO
FOR NONLINEAR EQUATIONS
INVOLVING $\mathcal{M}$ - DERIVATIVES

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A Newton - Kantorovic's process for solving nonlinear equations with an emphasis or nonlinearity of $\mathcal{M}$ - derivatives has been investigated.

1. INTRODUCTION

In [1], A. Donescu has introduced concepts of $\mathcal{M}$ - linearity, $\mathcal{M}$ - differentiability of map and studied the following modified Newton - Kantorovic's method:

$$x_{n+1} = x_n - \Gamma_{zn} F(x_n),$$

where Γ_{zn} are right inverses of the $\mathcal{M}$ - derivatives $F'(z_n)$. A convergence theorem for process under a severe restriction that there exits an uniformly bounded inverse Γ_x for almost every some neighbourhood of the initial approximation x_0 has been proved.

Since $F'(z_n)$, and therefore Γ_{zn} are nonlinear, it will be more justified to consider the pro

$$x_{n+1} = x_n + \Gamma_{zn}[-F(x_n)].$$

Clearly that the Donescu's convergence theorems remain valid for the last method. How in this note, we are particularly interested in the scalar case $X = Y = R^1$ for process (1.2), w to some extend, the convergence results are stronger and the conditions are easier to verify.

We begin by recalling some concepts and results which will be frequently used in this The reader should be refered to the work [1] for details.

Let X, Y, Z be real linear normed spaces. A mapping $T : X \to Y$ is $\mathcal{M}$ - linear, if T is Lips continuous and positively homogeneous. The set $\mathcal{ML}(X,Y)$ of all $\mathcal{M}$ - linear mappings, end with a standard linear structure and a norm defined by $\|T\| = \sup_{x_1 \neq x_2} \|T(x_1) - T(x_2)\|/\|$ $x_2\|$ becomes a linear normed space. Moreover, $\mathcal{ML}(X,Y)$ is a Banach space if Y is a Ba space. If $T_1 \in \mathcal{ML}(X,Y)$, $T_2 \in \mathcal{ML}(Y,Z)$, then $T_2T_1 \in \mathcal{ML}(X,Z)$ and $\|T_2T_1\| \leq \|T_2\|\,\|T_1\|$. $T_1, T \in \mathcal{ML}(X,Y)$ and suppose that T has an inverse $T^{-1} \in \mathcal{ML}(Y,X)$. Further, assume that $T^{-1}T_1\| \leq k < 1$, where I denotes the identity operator in X. then there exists $T_1^{-1} \in \mathcal{ML}($ and $\|T_1^{-1}\| \leq \|T^{-1}\|/(1-k)$.

A mapping $F : X \to Y$ is $\mathcal{M}$ - differentiable at $x_0 \in X$ if there exists a $\mathcal{M}$ - linear map $T \in \mathcal{ML}(X,Y)$ such that $\|F(x_0 + h) - F(x_0) - T(h)\| = O(\|h\|)$ for all h $h \in X$. The mappi is called a $\mathcal{M}$ - derivative of F at X_0 and is denoted as $T := F'(x_0)$ Note that $\mathcal{M}$ - derivative exists, is unique.

et F be $\mathcal{M}$ - differentiable and the mapping $x \mapsto F'(x)h$ is continuous on the segment $[x_0, x_0+$ $\{x = x_0 + th : 0 \le t \le 1\}$, with the possible exception of a countable set, then there holds a -value formula:

$$F(x_0 + h) - F(x_0) = \int_0^1 F'(x_0 + th)(h)dt.$$

2. THE NEWTON - KANTOROVIC'S METHOD

Consider an operator equation:

$$F(x) = 0 \tag{2.1}$$

$F. : X \to X$ is a $\mathcal{M}$ - differentiable mapping and X is a real Banach space. Suppose that -th approximate solution x_n is found (x_0 is given). Due to the $\mathcal{M}$ - differentiability of F at e have $F(x) = F(x_n) + F'(x_n)(x - x_n) + O(\|x - x_n\|)$. Neglecting the infinitesimal term, we the next approximation x_{n+1} from a $\mathcal{M}$ - linear equation: $F(x_n) + F'(x_n)(x_{n+1} - x_n) = 0$. $x_n)$ processes an inverse $[F'(x_n)]^{-1} \in \mathcal{ML}(X)$, then:

$$x_{n+1} = x_n + \Gamma_n(-F_n), \tag{2.2}$$

for simplifying notation, we put $\mathcal{ML}(X) := \mathcal{ML}(X, X)$,

$$F_n := F(x_n); \; F'_n := F'(x_n); \; \Gamma_n := F_n'^{-1}.$$

A simple Newton's method is defined by:

$$x_{n+1} = x_n + \Gamma_0(-F_n).$$

Generally speaking Γ_n and Γ_0 are not homogeneous, therefore $\Gamma_n(-F_n) \neq -\Gamma_n F_n$, $\Gamma_0(-F_n) \neq$ F_n.

rem 2.1. *Let F be $\mathcal{M}$ - differentiable on a closed ball $\overline{B} := \overline{B}(x_0, r)$ and its $\mathcal{M}$ - derivative be hitz-continuous on $\overline{B}$ with the possible exception of a countable set: $\|F'(x) - F'(y)\| \le L\|x - y\|$ $\in \overline{B})$. Further, suppose that F'_0 possesses an inverse $\Gamma_0 \in \mathcal{ML}(X)$ such that:*

$$\forall u, v \in -F(\overline{B}) \; \|\Gamma_0(u) - \Gamma_0(v) - \Gamma_0(u - v)\| \le \varepsilon \|u - v\|. \tag{2.4}$$

f there hold the follow relations:

$$\|\Gamma_0\| \le M; \; \|F'_0\| \le N; \; \|F_0\| \le \eta; \; h := M^2 L\eta/2 < 1/4. \tag{2.5}$$

$$M\eta(1 - \sqrt{1 - 4h})/2 < r. \tag{2.6}$$

$$0 < \varepsilon < M\sqrt{1 - 4h}/(MN + 1 - \sqrt{1 - 4h}). \tag{2.7}$$

there exits a solution $x^* \in \overline{B}_0 = \overline{B}(x_0, M\eta t_0)$ of (2.1), where t_0 is the smallest positive n of the equation $ht^2 - t + 1 = 0$. Moreover, the simple Newton's method (2.3) is convergent ere holds an error estimate: $\|x_n - x^*\| \le cq^n$, where $q = q_1 + \varepsilon(N + q_1/M)$, $q_1 := 1 - \sqrt{1 - 4h}$, nst > 0.

The proof is proceeded as in [2-4] with some obvious modifications. First note that from it follows $\overline{B}_0 \subset \overline{B}$.

et $A(x) := x + \Gamma_0(-F(x))$. Using the $\mathcal{M}$ - linearity of Γ_0, the Lipschitz-continuity of $F(x)$ on formula (1.3), we find:

$$\begin{aligned}
\forall x \in \overline{B}_0\,,\ \|A(x) - x_0\| &= \|x - x_0 + \Gamma_0[-F(x)]\| \\
&= \|\Gamma_0[-F(x)] - \Gamma_0 F_0'(x_0 - x)\| \le M\,\| - F(x) - F_0'(x_0 - x)\| \\
&\le M\,\{\eta + \|F_0 - F(x) - F_0'(x_0 - x)\|\} \\
&= M\,\{\eta + \|\int_0^1 [F'(x + t(x_0 - x)) - F_0'](x_0 - x)dt\|\} \\
&\le M\,\{\eta + L\|x_0 - x\|^2/2\} \le M\,\{\eta + L(M\eta t_0)^2/2\} = M\eta t_0\,.
\end{aligned}$$

Thus a maps $\overline{B}_0$ into itself.

Further, for all $x,\ y \in \overline{B}_0$ we have:

$$\begin{aligned}
\|A(x) - A(y)\| \le \| &- \Gamma_0 F_0'(y - x) + \Gamma_0[F(y) - F(x)]\| \\
&+ \|\Gamma_0[-F(x)] - \Gamma_0[-F(y)] - \Gamma_0[F(y) - F(x)]\|\,.
\end{aligned}$$

Again, using (1.3) and taking into account the $\mathcal{M}$ - linearity of Γ_0 and the Lipschitz-cont
of $F'(x)$ we get:

$$\begin{aligned}
\| - \Gamma_0 F_0'(y - x) + \Gamma_0[F(y) - F(x)]\| &\le M \int_0^1 \|F'(x + t(y - x)) - F_0\|\,\|y - x\|\,dt\,. \\
&\le ML(M\eta t_0)\,\|y - x\|\,.
\end{aligned}$$

From (2.4) it follows that

$$\begin{aligned}
&\|\Gamma_0[-F(x)] - \Gamma_0[-F(y)] - \Gamma_0[F(y) - F(x)]\| \\
&\le \varepsilon\,\|F(y) - F(x)\| \le \varepsilon \int_0^1 \|F'(x + t(y - x))\|\,\|y - x\|\,dt \\
&\le \varepsilon \int_0^1 \|F'(x + t(y - x)) - F'(x_0)\|\,\|y - x\|\,dt + \varepsilon\,\|F_0'\|\,\|y - x\|\,.
\end{aligned}$$

Thus,

$$\|\Gamma_0[-F(x)] - \Gamma_0[-F(y)] - \Gamma_0[F(y) - F(x)]\| \le \varepsilon[L(M\eta t_0) + N]\,\|y - x\|\,.$$

Combining inequalities (2.8) - (2.10) we arrive at:

$$\|A(x) - A(y)\| \le [(M + \varepsilon)ML\eta t_0 + \varepsilon N]\,\|y - x\| = q\,\|y - x\|\,.$$

Condition (2.7) ensures that $q \in (0,\,1)$. The conclusion of Theorem 2.1 now follows im
ately from the Banach fixed-point theorem.

Remark 2.1. If $F(x)$ is Frechet - differentiable at x_0 and $F'(x_0)$ has a bounded inverse I
conditions (2.4) and (2.7) are automatically fulfilled.

Example 2.1. [1]. Consider a mapping $F : (x_1,\,x_2) \in R^2 \mapsto (F_1(x),\,F_2(x)) \in R^2$ defin
$F_2(x) = |x_2|$,

$$F_1(x) = \begin{cases} x_1^2 + x_1 - 2 + x_2 & \text{if } x_1 < 1 \\ x_1 - 1 - x_2 & \text{if } x_1 \ge 1 \end{cases}$$

Clearly that $x^* = (1; 0)$ is a unique solution of (2.1). Further, F is not Frechet - differentiable ıy neighbourhood of x^*. On the other hand, F is Frechet - differentiable at $x_0 = (1.5; 0.5)$. ying (2.3), we get $x_1 = x_0 + \Gamma_0(-F_0) = (1; 0) = x^*$.

In the remainder of this section we consider the Banach space X with the so-called (α) - erty, i.e:

(α) $\quad \forall A, B, C \in \mathcal{ML}(X)\ \|AB - AC\| \le \|A\|\,\|B - C\|$.

position 2.1.

(i) $T \in \mathcal{ML}(R^1)$ *iff* T *is of the form:*

$$T(x) = \begin{cases} \alpha x & x \ge 0 \\ \beta x & x < 0, \end{cases} \tag{2.11}$$

e $\alpha, \beta \in R^1$ *are constants. Moreover,*

$$\|T\| = \max\{|\alpha|, |\beta|\}.$$

(ii) T *has an inverse* $T^{-1} \in \mathcal{ML}(R^1)$ *iff* $\alpha\beta > 0$.

f. (i) Let $T \in \mathcal{ML}(R^1)$ then for all $x \ge 0$ $T(x) = xT(1) = \alpha x$, where $\alpha = T(1)$; similarly, < 0, $T(x) = (-x)T(-1) = \beta x$ with $\beta := -T(-1)$. Now suppose that T is defined by (2.11). rly that T is positively homegenous, moreover, $|T(x) - T(y)| \le M|x - y|$ for all $x, y \in R^1$, e $M = \max(|\alpha|, |\beta|)$.

Thus $T \in \mathcal{ML}(R^1)$ and $\|T\| \le M$. Since $|T(1) - T(0)|/|1 - 0| = |\alpha|$, $|T(-1) - T(0)|/|-1-0| =$ t follows that $\|T\| = M$.

(ii) Assume for the definiteness that $\alpha, \beta > 0$. Then the equation $T(x) = y$ has a unique ion:

$$x = T^{-1}(y) = \begin{cases} y/\alpha & y \ge 0 \\ y/\beta & y < 0. \end{cases}$$

Simillarly, if $\alpha, \beta < 0$, then

$$x = T^{-1}(y) = \begin{cases} y/\beta & y \ge 0 \\ y/\alpha & y < 0. \end{cases}$$

position 2.2. $(R^1, |.|)$ *possesses the* (α) *- property.*

f. Suppose that $T \in \mathcal{ML}(R^1)$ and $|Tx| \le \lambda|x|$ for all $x \in R^1$ then it is easy to show that $\le \lambda$. Now let $A, B, C \in \mathcal{ML}(R^1)$ then for any $x \in R^1$ $|(AB - AC)x| = |ABx - ACx| \le$ $|Bx - Cx| = \|A\|\,|(B - C)x| \le \|A\|\,\|B - C\|\,|x|$, there fore $\|AB - AC\| \le \|A\|\,\|B - C\|$.

Note that the (α) - property has been used for proving the local and semilocal convergence ems for process (2.2).

Since these convergence theorems can be stated and proved as those given in [2-4], they will nitted.

3. SCALAR CASE $X = R^1$

Suppose that f is $\mathcal{M}$ - differentiable on $[a, b] \in R^1$. Then

$$f'(x)h = \begin{cases} \alpha(x)h; \ h \geq 0 \\ \beta(x)h; \ h < 0, \end{cases}$$

where $\alpha(x), \beta(x) \in R^1$.

We say that $f'(x)$ does not change sign on $[a, b]$ if $\operatorname{Sgn}\alpha(x) = \operatorname{Sgn}\beta(x) = \text{const}\ \forall x \in [a,$

Proposition 3.1. *If $f'(x) < 0$ ($f'(x) > 0$) on $[a, b]$ then $f(x)$ is decreasing (increasing).*

Proof. Suppose $f'(x) < 0$, then for any $x < y$, from the mean-value formula (1.3), we h $f(x) - f(y) = \int_0^1 f'(y + t(x-y))(x-y)dt = \int_0^1 \beta(y + t(x-y))(x-y)dt > 0$. Thus $f(x) >$ hence $f(x)$ is decreasing.

Proposition 3.2. *Let f be convex and differentiable on $[a, b]$, then:*

$$f(x) - f(y) \geq f'(y)(x-y)\ \forall x, y \in [a, b]$$

Proof. Using the convexity and $\mathcal{M}$-differentiability of f, for fixed $x, y \in [a, b]$ and any $t \in ($ we have $tf(x) + (1-t)f(y) \geq f(tx + (1-t)y) = f(y + t(x-y)) = f(y) + f'(y)(t(x-y)) + o(|x$ $= f(y) + tf'(y)(x-y) + o(t)$. It follows that $f(x) - f(y) \geq f'(y)(x-y) + o(t)/t$. Letting $t \to$ we come to (3.1).

Now we consider equation (2.1), where the scalar function $f(x)$ is convex or concave its $\mathcal{M}$-derivative does not change the sign. We assume that $f(x)$ is convex and $f'(x) < 0$, remaining cases can be considered similarly.

Theorem 3.1. *Let $f(x)$ be convex and $\mathcal{M}$-differentiable on $[a, b]$. Suppose that $|f'(x)| \leq M$ $f'(x) < 0\ \forall x \in [a, b]$.*

Further assume that x^* is solution of (2.1). Then the Newton-Kantorovic's method starting from any $x_0 \in [a, b]$ such that $f(x_0) \geq 0$, is convergent.

Proof. We prove by induction the following relations

$$f(x_n) \geq 0$$
$$x_n \leq x_{n+1}$$
$$x_n \leq x^*.$$

First we note that (3.3), (3.4) are implied from (3.2). Indeed $x_{n+1} = x_n + f_n'^{-1}[-f$ $x_n - f_n/\alpha_n \geq x_n$. Further, if $x_n > x^*$ then by proposition 3.1, $0 \leq f(x_n) < f(x^*) = 0$ whi impossible. Therefore $x_n \leq x^*$.

Now suppose that for all $0 \leq k \leq n$, $f(x_k) \geq 0$. Using proposition 3.2, we get $f_{n+1} -$ $f_n'(x_{n+1} - x_n) = f_n' f_n'^{-1}[-f_n]$, or $f_{n+1} \geq 0$. Thus all relations (3.2) - (3.4) are satisfied. (3.3), (3.4) it follows that the sequence x_n is convergent. Let $\lim_{n\to\infty} x_n = \xi \leq x^*$. Since $| -$ $|f_n'(x_{n+1} - x_n)| \leq M|x_{n+1} - x_n|$ we get $f(\xi) = 0$. Suppose $\xi \leq x^*$ then $0 = f(\xi) > f(x^*)$ This contradiction shows that $\xi = x^*$. If for all $x \in [a, b]$, $|f(x)| \geq m > 0$ then we have an estimate $0 \leq x^* - x_n \leq f(x_n)/m$.

The following simple example shows that method (2.2) is more rational than the "traditi Newton-Kantorovic's method.

$$x_{n+1} = x_n - f_n'^{-1} f_n$$

case of $\mathcal{M}$ - differentiable mappings.

ple 3.1. Let

$$f(x) = \begin{cases} -15x + 2 & \text{if } -1 \le x \le 0 \\ -10x + 2 & \text{if } 0 \le x \le 0.1 \\ -5x + 1.5 & \text{if } 0.1 \le x \le 0.2 \\ -2.5x + 1 & \text{if } 0.2 \le x \le 1. \end{cases}$$

Clearly that f is convex and $\mathcal{M}$ - differentiable on $[-1, 1]$.

Note that $f(0)f(1) < 0$ then $f(x)$ has a solution $x^* \in (0, 1)$. Further, $f'(x) < 0$ for all $-1, 1]$. Starting from $x_0 = 0$ $(f(x_0) = 2 > 0)$ we need 5 iterations (3.5) or only 3 iterations to achieve $x^* = 0.4$.

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VÀI NHẬN XÉT VỀ PHƯƠNG PHÁP NEWTON - KANTOROVICH GIẢI PHƯƠNG TRÌNH VỚI $\mathcal{M}$ - ĐẠO HÀM

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Báo cáo trình bày một cải biên của phương pháp Newton - Kantorovich giải phương trình với tử không trơn. Thuật toán cải biên này tỏ ra hợp lý hơn phương pháp Newton - Kantorovich biết, vì nó đã tính đến tính không thuần nhất của $\mathcal{M}$ - đạo hàm.