ON THE CONDITIONS UNDER WHICH Sub(L) DETERMINES L UP TO ISOMORPHISM

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1. INTRODUCTION

"Find conditions on a lattice L under which Sub(L) determines L up to isomorphism" i of the basic and important topics in studying the lattice Sub(L) (Grätzer's problem [1]).

In [2] Hoang Minh Chuong has given some conditions under which Sub(L) determines to isomorphism or dual isomorphism.

Studying the problem, in [3, 4] we have proposed the concept of contractible sublattice proved "Let L be a lattice having no contractible sublattices. Then Sub(L) determines L isomorphism or dual isomorphism".

In this paper, by contractible sublattice method and the above mentioned result we will some classes of lattices satisfying Grätzer's problem, that is, class of Boolean lattices and cl the free lattices.

2. RESULTS

First, we recall some concepts and known results.

We say that Sub(L) determines L up to isomorphism if: for an arbitrary lattice L', Sub(L) implies $L \cong L'$. In connection with this in [5] there has already been proved:

Theorem (I). *Let L, L' be arbitrary lattices. Then Sub(L)≅ Sub(L') if and only if there ex square preserving bijection $\varphi : L \to L'$.*

In [3, 4] we also have the following.

Definition (II). A proper sublattice A of a lattice L with $|A| > 1$ is called a contractible subl if A satisfies the following conditions:

(a) A is a convex sublattice.

(b) If $\langle a, b; c, d\rangle$ is a square in L then $c \in A \Leftrightarrow d \in A$.

Theorem (III). *Let L be a lattice having no contractible sublattices. Then Sub(L) determi up to isomorphism of dual isomorphism.*

Here we will study a concept, which will be needed for our problem.

Definition 2.1. A lattice L is called totally symmetric if L is isomorphic to its dual lattice

tement 2.2. *If Sub(L) determines L up to isomorphism then L is totally symmetric.*

of. Consider the map: $j : L \to L^*$, $j(a) = a$ and $a \leq b \Leftrightarrow j(a) \geq j(b)$ $\forall a, b \in L$. Clearly j is uare preserving bijection. By Theorem (I) we have $\text{Sub}(L) \cong \text{Sub}(L^*)$ and by the assumptions iave $L \cong L^*$, which was to be proved.

:e. The converse statement to (2.2) is not true, or in :r words, the total symmetry is not sufficient for L to letermined by $\text{Sub}(L)$ up to isomorphism. There exist ices L which are totally symmetric but they are not :rmined by $\text{Sub}(L)$ up to an isomorphism. For such a pose, let us consider the lattice L in Fig. 1.

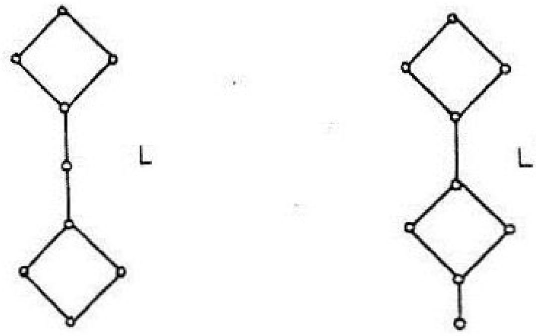

Fig. 1

This lattice is totally symmetric, but it is not isomor- : to L'. However, it is easy to point out a square pre- ing bijection $\varphi : L \to L'$ and thus, $\text{Sub}(L) \cong \text{Sub}(L')$ .Theorem (I)).

Now, we consider the lattices which have no contractible sublattices and are totally symmetric. ılying Theorem (III) we have the following.

position 2.3. *Let L be a totally symmetric lattice having no contractible sublattices. Then (L) determines L up to isomorphism.*

In the sequel, by Proposition 2.3 we will show some types of lattices satisfying Grätzer's blem.

First, we consider the *Boolean lattices*. A distributive lattice B containing 0 and 1 is called ılean if $\forall a \in B$, $\exists c \in B$ (which is called a complement of a) such that $a \wedge c = 0$, $a \vee c = 1$.

position 2.4. *If B is a Boolean lattice then Sub(B) determines B up to isomorphism.*

of. (i) We prove that B has no contractible sublattices. By contradiction assume that A is a tractible sublattice of B. Clearly $0, 1 \notin A$. Since $|A| > 1$, there exist $a, b \in A$ such that $a < b$. :e $c \in B$ as a complement of a, i.e. $a \wedge c = 0$, $a \vee c = 1$.

Consider the element $x = b \wedge c$. Since $a \vee x = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$ we have $a \| x$ (a omparable with x). Applying Definition (II) to the square $\langle a, x; a \wedge x, b\rangle$ we have $0 = a \wedge x \in A$, this is desired contradiction. -

(ii) Now, we have only to prove that B is totally symmetric. We will use the fact that B is ributive and the complements in B are unique.

We define a map $\varphi : B \to B$ putting $\varphi(a) = a'$ $\forall a \in B$, where a' is the complement of a. viously φ is a bijection. We show that $a \leq b \Leftrightarrow \varphi(a) \geq \varphi(b)$.

Indeed, assume $a \leq b$ we have $a = a \wedge b$ and $b' = b' \wedge 1 = b' \wedge (a' \vee a) = (b' \wedge a') \vee (b' \wedge a) =$ $\wedge a') \vee (b' \wedge b \wedge a) = (b' \wedge a') \vee 0 = b' \wedge a'$, i.e. $b' \leq a'$.

Similarly, if $b' \leq a'$ we also have $a \leq b$.

In short, φ is a dual isomorphism.

Consider the map $j : B \to B^*$, $j(a) = a$ and $a \leq b \Leftrightarrow j(a) \geq j(b)$ $\forall a, b \in B$; It is a dual norphism. Thus, the composition of φ and j gives us a lattice isomorphism $j.\varphi : B \to B^*$.

The proof is completed.

Now we turn to the free lattices [6].

The free lattice $F = FL(X)$ with the set of the generators $X = \{x_i,\ i \in I\}$ is constructe follows:

(i) F consists of the "terms" which are defined by induction on their length:

a) $x_i,\ i \in I$ are the terms the length $d(x_i) = 0$.

b) If u, v are two terms such that $d(u) + d(v) = n - 1$ then $u \wedge v$, $u \vee v$ are terms $d(u \wedge v) = d(u \vee v) = n$.

(ii) The order "$\leq$" in F is defined basing on the following four principles:

(1) $p \vee q \leq a$ if $p \leq a$ and $q \leq a$,

(2) $p \vee q \leq a$ if $b \leq p$ and $b \leq q$,

(3) $p \wedge q \leq a$ if $p \leq a$ or $q \leq a$,

(4) $b \leq p \vee q$ if $b \leq p$ or $b \leq q$.

(iii) Put $\inf(u, v) = u \wedge v$ and $\sup(u, v) = u \vee v$. By the principles (1) $\rightarrow$ (4) in (ii) we F as a lattice.

(iv) Given and arbitrary lattice L and a map $f : X \rightarrow L$ we can always establish a la homomorphism $\varphi : F \rightarrow L$ such that $\varphi(x_i) = f(x_i)$, $i \in I$.

For later use in the demonstration of the following proposition 2.6, first we prove:

Lemma 2.5. *Let A be a contractible sublattice in L and $k \in L \setminus A$. If $\exists a \in A$ such that k then $k > x$, $\forall x \in A$.*

Proof. Assume that $k > a$ for some $a \in A$. Consider an arbitrary $x \in A$.

If $x > k$ then $k \in A$ due to convexity of A. It contradicts the fact that $k \notin A$.

If $x \| k$ then $x > x \wedge k > x \wedge a$. By Definition (II) we have $x \wedge k \in A$ and $x \vee k \in A$. F $x \vee k > k > a$ it follows that $k \in A$, but it is impossible.

Thus necessarily $k > x$, and the lemma is proved.

Proposition 2.6. *Let $F = FL(X)$ be a free lattice. Then $Sub(F)$ determines F up to isomorph*

Proof. (i) we prove that F has no contractible sublattices. The cases, where $|x| \leq 2$ are tri Consider F with $|X| > 2$. We argue by contradiction, supposing that F has a contractible subla A.

1) Consider the element $u \in A$. By induction on the length $d(u)$ we prove that: if the gener x is contained in u then $x \in A$.

If $d(u) = 0$, i.e. $u = x$ then it is obviously $x \in A$.

Now, assume $d(u) = n > 0$. We always have $u = p \wedge q$ or $u = p \vee q$ where $d(p)$, $d(q)$ From the square $\langle p, q;\ p \wedge q, p \vee q\rangle$ we deduce immediately p, $q \in A$ (see Definition (II)). Thus conclusion of part 1) now is followed directly from the induction hypothesis.

2) Using the above part 1) and since $|A| > 1$, we observe that there exist no less than terms with null-length x, $y \in A$.

Consider a generator $z \in X$, $z \neq x$, y and the square $\langle x, z;\ x \wedge z, x \vee z\rangle$. If $x \vee z \notin A$ $y < x \vee z$ (see Lemma 2.5). By Principle (4) in (ii) we have $y \leq x$ or $y \leq z$. But it does not

$\not\leq x$ and $y \neq z$. Thus $x \vee z \in A$ and therefore we have $z \in A$ (see Definition (II)). Because z trary we have $X \subseteq A$, i.e. $A = F$. This contradicts the assumption that A is contractible.

i) To demonstrate that F is totally symmetric ($F \cong F^*$) we shall use the universal property free lattice (see Part (iv)).

or the lattice F^* we take the embedding $f : X \to F^*$, $f(x_i) = x_i$, $i \in I$. Thus, there exists a homomorphism $\varphi : F \to F^*$ such that $\varphi(x_i) = f(x_i)$, $i \in I$. Evidently φ is an isomorphism.

he proposition is proved.

ıples. We represent some lattices L, which are determined by $\text{Sub}(L)$ up to isomorphism s the lattices mentioned in (2.4) and 2.6).

or the lattice in Fig. 2, all the ions of (2.3) are satisfied but it is r Boolean nor free.

n the other hand, the lattices in show that: there exist the lattices have contractible sublattices and the condition: "$\text{Sub}(L)$ determines o isomorphism".

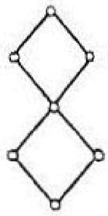

Fig. 2

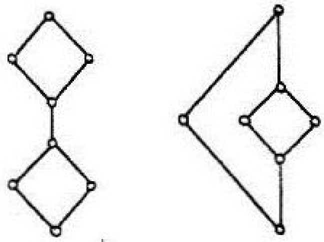

Fig. 3

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VỀ CÁC ĐIỀU KIỆN ĐỂ DÀN Sub(L) XÁC ĐỊNH DÀN L SAI KHÁC NHAU MỘT ĐẲNG CẤU

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ài báo này nghiên cứu bài toán Grätzer [1]: tìm điều kiện trên dàn L sao cho $\text{Sub}(L)$ xác , sai khác nhau một đẳng cấu.

ằng phương pháp dàn con co được chúng tôi đã đi đến một kết quả thú vị đối với bài toán ι. Chúng tôi đã chỉ ra được một số lớp các dàn L thỏa mãn điều kiện $\text{Sub}(L)$ xác định L ác nhau một đẳng cấu. Đó là lớp các dàn Boole và lớp các dàn tự do.