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OPTIMAL CONTROL PROBLEM FOR DIFFUSION PROCESSES WITH LONG RUN AVERAGE COST PER UNIT TIMES

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1. INTRODUCTION

n this paper, we study the control problem for dynamical systems whose state X_t at time t cribed by a stochastic differential equation of the form

$$\begin{cases} dX_t &= b(X_t, U_t)\, dt + \sigma(X_t)\, dW_t \\ X_0 &= x \in R^d \end{cases} \tag{1.1}$$

: (W_t) is a wiener process defined on a stochastic basis $(\Omega, \mathcal{F}_t, P)$, satisfying the standard tions and b, σ are given function.

\s usually the control process (U_t) is supposed to be progressively measurable with respect : filtration $\mathcal{F}_t$, with value in a metric compact space A. For any admissible control (U_t) we ler the solution $X_t^U(x)$ of (1.1) associated with the control (U_t) staring from x at $t = 0$. We ler the average cost of (U_t) per unit time

$$J(x, U) = \limsup_{T \to \infty} \frac{1}{T} E \int_0^T h(X_t, U_t)\, dt$$

hich to minimise the cost $J(x, U)$ over the class U_{ad}

This problem has been dealt with in various ways. In [7] admissible control are defined as ıaps from R^d into A satisfying the Lipschitz condition and under his assumption the author hown that there exists a Lipschitz feedback control. Kushner in [6] used relaxed control to problems for "wideband noise driven" with are "close" to a diffusion. In this article we use ıethods dealt with by [5] for discrete cases to develop these results. Under the elliptic and ıess hypotheses we show that there exists an optimal markovian measure. Hence using the ion theorem we can prove the existence of optimal markovian control process.

2. NOTATIONS AND HYPOTHESES

Hypotheses

,et A be a metric compact space, called the actions space and let

$$\begin{aligned} \sigma &: R^d \to d \times d - \text{matrices} \\ b &: R^d \times A \to R^d \\ h &: R^d \times A \to R \end{aligned} \tag{2.1}$$

Throughout this paper we suppose that the following hypotheses are satisfied:

Hypothesis 1: *σ, b, h are continuous functions in (x, a) and h is bounded ; σ locally Lip continuous .*

We consider the generator associated with the coefficients σ and b of the equation (1) by

$$L^a f(x) = \frac{1}{2}\sum a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum b_i(x, a)\frac{\partial f}{\partial x_i}$$

where a_{ij} is the ij^{th}- element of the matrix $\sigma\sigma^*(x)$ and $b(x) = (b_1, b_2, \ldots, b^d)^T$; $a \in A$. operator L defined on the space of twice continuous differential function with bounded deriva , say $C_b^2(R^d)$

Hypothesis 2: *The operator L^a is uniformly elliptic.* That this : there is a $\alpha > 0$ such tha

$$\sigma\sigma^*(x) \geq \alpha I \quad \text{for any } x \in R^d$$

2.2. Admissible control

We define admissible controls as the case of finite horizon. Let U be an admissible co then X_t^U is the solution of (1) associated with U if and only if

$$C_t(f, U) = f(X_t) - f(X_s) - \int_s^t L^{U_h} f(X_h)\, dh$$

is a P- martingale after s for any $f \in C_b^2(R^d)$. Therefore we have the following definition

Definition 1. An admissible control is a term $U^0 = (\Omega, \mathcal{F}_t, P, X_t, U_t, \nu)$ such that:

(i) $(\Omega, \mathcal{F}_t, P)$ is a probability space with the filtration $\mathcal{F}_t$.

(ii) U_t is a progressively measurable process with values in A.

(iii) (X_t) is a continuous process with values in R^d such that

$$C_t(f, U) = f(X_t) - f(X_s) - \int_0^t L^{U_h} f(X_h)\, dh$$

is a P- martingale for any $f \in C_b^2(R^d)$

(iv). X_0 has the distribution ν.

The set of admissible controls starting from ν at $t = 0$ is denoted by $\mathcal{A}_0$. Let $U^0 \in \mathcal{A}_0$ consider the payoff

$$J_0(\nu, U^0) = \limsup_{T \to \infty} \frac{1}{T} P \int_0^T h(X_t, U_t) dt$$

where $P(\Phi) = \int \Phi\, dP$. Our aim is to minimise $J_0(\nu, U)$ over class $\mathcal{A}^0(\nu)$

2.3. Relaxed controls

We write for V the space of generalised actions consisting of random measures on $R^+ \times$ the form $dt.q(t, da)$ where dt is the Lebesgue measure on R^+ and $q(t, da)$ is a probability any $t \in R^+$. With the vague topology, V is a metric compact space. This is the same as the ·le topology on $R^+ \times A$ (see [3]).

We write, from now on, $f(t, q_t)$ for $\int_A f(t,a)\, q_t(da)$ for any measurable f. The space V ndowed with its Borel $\sigma-$ field $\mathcal{V}$ which is also the smallest $\sigma-$ field such that the maps · $\int_{R^+} \int_A f(s,a)\, q(ds, da)$ are measurable for any measurable function f continuous in a with ıpact support. We also introduce the filtration $(\mathcal{V}_t)$ where $\mathcal{V}_t$ is the $\sigma-$ algebra generated by measures $\{1_{[0,t]}\, q : q \in \mathcal{V}\}$. The details of these definitions can be refereed in [1].

finition 2. A relaxed control is a term $\mathcal{U} = (\Omega, \mathcal{F}_t, P, X_t, q, \nu)$ such that

(i) $(\Omega, \mathcal{F}_t, P)$ is a probability space with the filtration $\mathcal{F}_t$ satisfying the general hypotheses.

(ii) q is a $V-$ valued stochastic process, $\mathcal{F}_t-$ adapted.

(iii) (X_t) is a continuous process with values in R^d such that

$$C_t(f,U) = f(X_t) - f(X_0) - \int_0^t \int_A L^a f(X_h) q(h, da)\, dh \tag{2.5}$$

P- martingale for any $f \in C_b^2(R^d)$

(iv) The distribution of X_0 is ν.

The set of relaxed control with the initial distribution ν is denoted by $\mathcal{A}(\nu)$. For any $U \in \mathcal{A}(\nu)$ consider the payoff

$$J(\nu, U) = \limsup_{T\to\infty} \frac{1}{T} P \int_0^T h(t,a)\, q(t, da)\, ds$$

$$J(\nu) = \inf\{J(\nu, U) : U \in \mathcal{A}(\nu)\}\,; \quad J^* = \inf_\nu J(\nu)$$

write $J(x)$ for $J(\nu)$ whenever ν is Diract mass at x. The pair (ν^*, U^*) where $U^* \in \nu^*$ is said be minimum if $J(\nu^*, U^*) = J^*$, The term $(\Omega, \mathcal{F}_t, P, X_t, q)$ is called optimal if for any $x \in R^d$ relaxed control $U = (\Omega, \mathcal{F}_t, P, X_t, q, x)$ satisfies the relation

$$J(x) = J(x, U)$$

:he following we can see that the set of admissible controls $\mathcal{A}_0(\nu)$ is complete in the sense that

$$J^* = \inf_\nu \inf\{J(\nu, U) \,:\, U \in \mathcal{A}(\nu)\} = \inf_\nu \inf\{J_0(\nu, U^0) \,:\, U^0 \in \mathcal{A}_0\}$$

. Rules of Controls

Similar a in [1], we formulate the problem on the canonical space $\mathcal{X}$ consisting of continuous ctions from R^+ into R^d endowed with the topology of uniform convergence on every bounded erval. Let $\mathcal{X}_t$ be the natural right - continuous filtration and (X_t) be the canonical process ined on $\mathcal{X}$. We put

$$\overline{\mathcal{X}} = \mathcal{X} \times V; \quad \overline{\mathcal{X}_t} = \bigcap_{s>t} \mathcal{X}_s \oplus \mathcal{V}_s$$

finition 3. A control rule (more briefly : a rule) is a probability measure R on $\overline{\mathcal{X}}$ such that system $(\overline{\mathcal{X}}, \overline{\mathcal{X}_t}, R, X_t, q, \nu)$ is a relaxed control.

The set of rules starting from ν is denoted by $\mathcal{R}(\nu)$. In term of control rules, optimal rela control can be interpreted as a family of rules $\{R_x \in \mathcal{R}(x) : x \in R^d$ such that $J(x, R_x) = J(x)$ any $x \in R^d$.

We are now able to define markovian controls by two ways : either we consider them term $U = (\Omega, \mathcal{F}_t, P_x, X_t, q, x)$ such that $q = dt \times q(X_t, da)$ or as a family of rules $\{R_x : x \in$ such that the term $(\overline{X}, \overline{X_t}, X_t, R_x, x)$ is a family of homogeneous markovian process . We will s that two these definitions are equivalent.

Theorem 1. *The family of rules $\{R_x : x \in R^d\}$ is markovian iff there exists a valued - meas map q from R^d into $\mathcal{P}(A)$ and a relaxed control $(\Omega, \mathcal{F}_t, P, X_t, dt\, q(X_t, da))$ such that R_x is the of the couple $(X_t, dt \times q(t, da))$ under P given $X_0 = x$.*

Proof. The sufficient condition is obvious. The necessary one is proved by a similar way as Theo 6.7 in [1] and we do not reproduce it here. ◇

In order to show the existence of optimal rules we follow the argument dealt with by Kur in [5] for the discrete case. First, we prove that there exists an optimal pair (ν^*, R^*) such tha is stationary distribution of X_t under R. After that by virtue of recurrent property of (X_t) we show that this rule is an optimal markovian control .

Let $U \in \mathcal{A}(\nu)$ be a relaxed control we put

$$\gamma_T^U(D) = \frac{1}{T} P \int_0^T 1_D(X_t)\, dt \quad D \in \mathcal{B}(R^d), \quad T > 0.$$

Hypothesis 3. *For any ν, the family*

$$\{\gamma_T^U(\cdot) : T > 0\,, U \in \mathcal{A}(\nu)\} \qquad ($$

is tight.

Hypothesis 3 will be true if we consider reflected diffusion processes in a bounded domai R^d (see [7]) or if the following hypothesis 3' is satisfied. Set

$$\Phi(x) = \sum_{i,j=1}^{d} a_{ij}(x) x_i x_j \; ; \; \Psi^a(x) = \frac{1}{\Phi(x)} \left[\frac{1}{2} \sum_{ij} a_{ij}(x) + \sum_{i=1}^{d} b_i(x, a) x_i \right]$$

and

$$\Phi^+(r) = \sup_{|x| = \sqrt{2r}} \Phi(x) \; ; \; \Psi^+(r) = \sup_{|x| = \sqrt{2r}} \sup_{a \in A} \Psi^a(x)$$

It is easy to check that $\Phi^+ > 0$ is locally Lipschitz continuous and Ψ^+ is measurable (see [2] 374). Hence there exists the minimal diffusion process ξ_t generated by the operator

$$L^+ = \Phi^+(r) [\frac{d^2}{dr^2} + \Psi^+(r) . \frac{d}{dr}]$$

Hypothesis 3': *The family $\{\sigma_T^\xi(\cdot) ; T > 0\}$ where*

$$\sigma_t^\xi(B) = \frac{1}{T} \int_0^T P(\xi_t \in B) \; ; \; T > 0$$

t.

ndeed, let $(\Omega, \mathcal{F}_t, P, X_t, q)$ be an arbitrary relaxed control. We put $H(t) = \int_0^t \frac{\Phi(X_s)}{\Phi+(|X_s|)}\, ds$ then asy to verify that $\frac{\lambda}{\Lambda} t \leq H(t) \leq t$ where λ and Λ respectively is the minimum of smallest naximum of biggest eigenvalues of the matrix $\sigma\sigma^*(x)$ in x. Moreover , $H(t)$ is a increasing ion. We write for $G(t)$ its inverse function. By the comparison theorems (see [2], pp 371; we have

$$|X(G(t))|^2 \leq \xi_t \quad \text{a.s}$$

e other hand

$$\int_0^T 1_{\{\xi_t > c\}}\, dt \geq \int_0^T 1_{\{|X(t)|^2 > c\}} dt = \int_0^{G(T)} 1_{\{|X(u)|^2 > c\}} H'(u) du$$
$$\geq \frac{\lambda}{\Lambda} \int_0^{G(T)} 1_{\{|X(u)|^2 > c\}} du$$

implies that

$$\frac{1}{T} \int_0^T 1_{\{\xi_t > c\}}\, dt \geq \int_0^{\frac{\Lambda}{\lambda}} 1_{\{|X(u)|^2 > c\}}\, dt$$

ypothesis 3', $\{\gamma_T^U(\cdot), T > 0\}$ is tight.

rem 2. *(see [5] Lemma 2.1) . For any initial distribution ν and for any $R \in \mathcal{R}(\nu)$, there a probability measure μ on $R^d \times A$ such that*

$$\int h(x,a)\, \mu(dx, da) \leq J(\nu, R) \tag{2.7}$$

$$\int_{R^d} \int_A L^a f(x) \mu(dx, da) = 0 \tag{2.8}$$

ny $f \in C_b^2$

f. Let ν and $R \in \mathcal{R}(\nu)$ arbitrary. We put

$$\gamma_T^R(D) = \frac{1}{T} R \int_0^T 1_D(X_t, q_t) dt \; ; \; D \in \mathcal{B}(R^d \times A)$$

$\{\gamma_T^R : T > 0\}$ is tight then there exists a sequence $(T_n) \to \infty$ such that

$$\gamma_{T_n}^R(\cdot) \overset{\text{weak}}{\Rightarrow} \mu(\cdot)$$

h is a continuous function then

$$\int h(x,a) \mu(dx, da) = \lim_{n \to \infty} \int h(x,a) \gamma_{T_n}^R(dx, da) \leq \limsup \frac{1}{T} R \int_0^T h(X_t, q_t)$$

he other hand, for any $f \in C_b^2$ we have

$$C_t(f, U) = f(X_t) - f(X_s) - \int_s^t L^{U_h} f(X_h)\, dh$$

P- martingale then by the law of large numbers it follows that

$$0 = \lim \frac{1}{T_n} E \int_0^{T_n} Lf(X_t, q_t) dt = \int L^a f(x) \mu(dx, da)$$

so we get (2.8) ◇

Lemma 2. *If $\{\mu_n\}$ is a sequence of probability measures satisfying Condition (2.8) then $\{\mu_n\}$ tight.*

Proof. Since A is compact then the sequence of projections of $\{\mu_n\}$ on A is automatically p compact. Suppose that $\mu_n(dx, da) = \mu_n(dx)\nu_n(x, da)$ and $\bar{b}_n(x) = \int b(x, a)\, \nu_n(x, da)$. Let X_n be the solution of the equation

$$\begin{aligned} dX_n(t) &= \bar{b}_n(X_n(t))\, dt + \sigma(X_n(t))dW_t \\ X_n(0) &= 0 \in R^d \end{aligned} \qquad (2$$

Then $U^n = (\Omega, \mathcal{F}_t, P, X_n(t), \nu_n(X_n, da), 0\ )$ is an relaxed control and $U^n \in \mathcal{A}(\delta_0)$. On the ot hand, from (2.3) and (2.8) it follows that μ_n is the unique invariant measure of Equation (2.9)

$$\mu_n(D) = \lim \frac{1}{T} P \int_0^T 1_D(X_n(t))\, dt$$

for any set D good enough. By Hypothesis 3, the sequence $\{\gamma_T^{U_n}(\cdot) = \frac{1}{T} P \int_0^T 1_D(X_n(t))\, dt\ T$ $0; n > 0\}$. is tight then it follows the tightness of $\{\mu_n\}$. The proof is complete. ◇

Lemma 3: *there exists a probability measure ν^* and a markovian control rule R^* such that*

$$J(\nu^*, R^*) = J^*$$

i.e, the pair (ν^*, R^*) is optimal.

Proof. Let $\{\nu_n, R_n\}$ be a sequence minimising $J(\cdot, \cdot)$. This means that

$$\lim_{n\to\infty} J(\nu_n, R_n) = J^*$$

By Lemma 2, for any n there exists a probability measure on $R^d \times A$, namely $\mu_n(\cdot)$, such tha

$$\int h(x, a)\, \mu_n(dx, da) \leq J(\nu_n, R_n)$$

From Lamma 2, the sequence $\{\mu_n\}$ is tight. Then there is a probability measure μ on $R^d \times A$ a sequence $(n_k) \to \infty$ such that

$$\mu_{n_k} \overset{weak}{\Rightarrow} \mu$$

Hence, we have

$$\int h(x, a)\mu(dx, da) = \lim_{n_k\to\infty} \int h(x, a)\mu_{n_k}(dx, da) \leq \liminf_{n_k\to\infty} J(\nu_{n_k}, R_{n_k}) = J^*$$

and

$$\int L^a f(x)\, \mu(dx, da) = 0$$

Suppose that $\mu(dx, da) = \nu(dx) \cdot q(x, da)$. We put

$$b^q(x) = \int b(x, a)\, q(x, da)$$

the equation

$$dX_t = b^q(X_t)\,dt + \sigma(X_t)dW_t \tag{2.10}$$

s a unique stationary solution with the stationary distribution $\nu(dx)$ It is easy to check that air (ν, R_ν is optimal where R_ν is the law of $(X_\cdot, dt \cdot q(X_\cdot, da)$ on $\overline{\mathcal{X}}$. The result follows. ◇

Since the diffusion matrix $a(x) = \sigma(x)\sigma^*(x)$ is non-degenerate then the invariant measure ν is lutely continuous with respect to the Lebesgue measure on R^d and support $\nu = R^d$ (see [AK]) the limit

$$\lim_{T\to\infty} \frac{1}{T} E \int_0^T \int_A h(X_t, a) q(X_t, da)\,dt = \int h(x,a)q(x,da)\nu(dx) = J^*$$

s for any $x \in R^d$ a.s (see [7] Th. 6.1). This means that $q(x, da)$ is a markovian optimal rol.

Denoting $A(x)f = \frac{1}{2}\sum a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $< b^a, \nabla f >= \sum b(x,a)\frac{\partial f}{\partial x_i}$, we now consider the equa-

$$A(x)\Phi + < b^q, \nabla\Phi > + h^q(x) = J^* \tag{2.11}$$

equation has a generalised solution Φ since σ is non degenerate. Let

$$s(x) = \{a \in A : A(x)\Phi + < b^a(x), \nabla\Phi > + h(x,a) \le J^*$$

2.10) $s(x) \neq \emptyset$ and $q(x, s(x)) \neq 0$ for any $x \in R^d$ Hence there is a measurable selection $R^d \to A$ such that

$$\frac{1}{2}\sum a_{ij}\frac{\partial^2\Phi}{\partial x_i \partial x_j} + \sum b_i(x, u^*(x))\frac{\partial\Phi}{\partial x_i} + h(x, u^*(x)) \le J^*$$

rtue of the generalized Ito's formula (see [4]) we have

$$E\Phi(X_t) = E\Phi(X_0) + E\int_0^t (A(x)\Phi + < b^{u^*}, \nabla\Phi >)ds$$
$$\le E\,\Phi(X_0) + E\int [J^* - h(X_s, u^*(X_s))]\,ds$$

inequality implies that

$$\limsup \frac{1}{T} E \int_0^T h(X_t, u(X_t))\,dt \le J^*$$

means that u^* is an optimal control process.

rem 3: *Under Hypotheses 1,2,3, there exists an optimal control process u^* which satisfies quation*

$$A(x)\Phi + < b^{u^*}, \nabla\Phi > + h(x, u^*(x)) \le J^*$$

From this result we can follow that the set A^0 is dense in A as we have mentioned. The proof y so we obmit it here.

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ĐIỀU KHIỂN TỐI ƯU CÁC QUÁ TRÌNH KHUẾCH TÁN VỚI GIÁ TRUNG BÌNH THEO THỜI GIAN

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Bài báo đề cập đến bài toán điều khiển tối ưu các quá trình khuếch tán với giá trung bì theo thời gian. Dưới giả thiết về tính compắc tương đối của lớp các điều khiển và sự không s biến của hệ số khuếch tán của quá trình tín hiệu, nhờ phương pháp tương tự như trong [5], chú tôi chỉ ra sự tồn tại của điều khiển tối ưu Markov. Bài báo là sự mở rộng phương pháp trong từ rời rạc lên trường hợp liên tục.