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ON THE EXISTENCE OF THE SOLUTION AND DU VARIATIONAL FORMULATION FOR THE OPTIMIZA OF THE DOMAIN IN ELLIPTIC PROBLEMS

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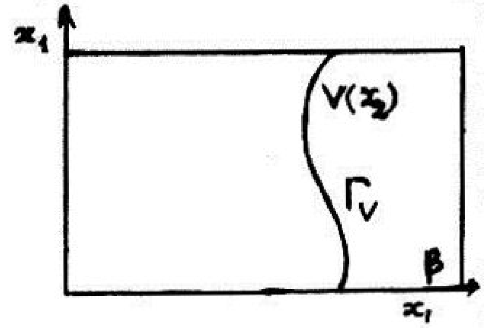
1. THE EXISTENCE OF THE SOLUTION OF THE PRIMARY PROBL

Let us consider the following problem: Let $\Omega(v) \subset R^2$ be the domain



$$\Omega(v) = \{(x_1, x_2) \mid 0 < x_2 < 1 \quad 0 < x_1 < v(x_2)\}$$

(see Fig.1)
where the function v is to be determined from problem $J_i(v) = \min_{w \in U_{ad}} J_i(w), i = 1, 2$.

$U_{ad} = \{w \in C^{(0),1}([0,1]), 0 < \alpha \leq w \leq \beta,$
$\quad |dw/dx_2| \leq C_1, \quad \int_0^1 w(x_2)dx_2 = C_2\}$

Here β, α, C_1, C_2 are given constants, $J_i(w) = J_i$
$i = 1, 2$ and

$$J_1(y(w)) = \int_{\Omega(w)} [y(w) - Z_0]^2 \, dx, \qquad Z_0 = \text{const (given)}$$

$$J_2(y(w)) = \int_{\Omega(w)} | \nabla y(w)|^2 \, dx,$$

and $y(w)$ is the solution of the following problem:

$$\begin{aligned} - \Delta y(w) &= f \quad && \text{in} \quad \Omega(w), \\ y(w) &= 0 && \text{on} \quad \Gamma w, \\ \partial y(w)/\partial n + \alpha_0 y(w) & && \text{on} \quad \partial\Omega(w) - \Gamma_N \equiv \Gamma_{N_w}, \end{aligned}$$

where $f \in L_2(\Omega_\beta)$ is given; $\Omega_\beta = (0, \beta) \times (0, 1)$, α_0 is a positive constant.

Let us set (the space of test functions):

$$V(w) = \{z \in H^1(\Omega(w)) \mid z = 0 \text{on} \quad \Gamma_w\}$$

A function $y(w) \in V(w)$ will be called a weak solution of the state problen (1.7) if

$$a(y, z) + A(y, z) = (f, z)_{0.\Omega(w)} \ \forall z \in V(w),$$

where
$$a(y, z) = (\nabla y, \nabla z)_{0.\Omega(w)},$$
$$A(y, z) = \alpha_0(y, z)_{0,\Gamma_{N_w}}$$

Using the trace theorem it can be shown that the bilinear form $a(y, z) +$. bounded on $V(w)$. On the other hand, the generalized Friedrichs inequality ens $V(w)$ - ellipticity of $a(y, z) + A(y, z)$. From this and the Lax-Milgram Theorem the e and uniqueness of $y(w)$ follows.

e shall prove that the optimization problem (1.2) is solvable for any of the cost
onals J_i, $i = 1, 2$.

em 1. The problem (1.2) has at least one solution for every cost functionals
1, 2.

roof: Let us consider a minimizing sequence $\{w_n\}$ such that $J_i(w_n) \to \inf_{w \in U_{ad}} J_i(w)$
$\to \infty$. Since the set U_{ad} is compact in $C([0,1])$ then by Arzela - Ascoli Theorem,
ay choose a subsequence, denoted again by (w_n) such that $w_n \to v$ in $C([0,1])$. We
$v \in U_{ad}$. Let $\Omega = \Omega(v)$ denote the domain determined by $\Gamma = \Gamma_v, \Gamma_N = \Gamma_{Nv}$.
urthermore, let Ω_n be domain bounded by the graph of the function w_n and let
the corresponding solution of the state problem (1.8), where $\Omega(w_n) = \Omega_n$. $\Gamma_{w_n} =$
$_{Nw_n} = \Gamma_{N_n}$ and $V(w_m)$ is determined respectively. It is readily seen from (1.8) that

$$\int_{\Omega_n} |\nabla y_n(w)|^2 dx \leq \int_{\Omega_n} f y_n dx \leq \|f\|_{0,\Omega\beta} \cdot \|y\|_{0,\Omega_n} \tag{1.9}$$

ng use of the generalized Friedrichs inequality we obtain

$$\|y\|^2_{1,\Omega_n} \leq \int_{\Omega_n} C|\nabla y(w)|^2 dx \tag{1.10}$$

C does not depend on m.
ombining (1.9) and (1.10) we arrive at the following estimate

$$\|y_n\|_{1,\Omega_n} \leq C_0 \quad \forall n \tag{1.11}$$

C_0 is independent of n.
et us choose $\delta > \beta$ and consider the set

$$V_\delta = \{v \in H^1(\Omega_\delta) \mid v = 0 \quad \text{on} \quad \Gamma_\delta\}$$

$\Omega_\delta = (0, \delta) \times (0, 1)$, $\Gamma_\delta = \{(\delta, x_2) \mid \quad 0 \leq x_2 \leq 1\}$. Then there exists a constant $\nu(\delta) > 0$
that

$$\int_{\Omega_\delta} |\nabla v|^2 dx \geq \nu(\delta) \|v\|^2_{1,\Omega_\delta}$$

very n we extend y_n by zero into the whole domain Ω_δ. We then have

$$\|y\|_{1,\Omega_\delta} = \|y\|_{1,\Omega_n} \leq C_0 \quad \forall n,$$

efore, there exist a subsequence of (y_n), denoted by y_n again, and $y^* \in V_\delta$ such that
y^* (weakly) in V_δ.
et $y^* \neq 0$ on the set $E \subset \Omega_\delta \setminus \bar{\Omega}$, $mes(E) > 0$. Let Ω_ϵ be the domain define by the
of $V + \epsilon$. Obviously, there exists $\epsilon > 0$ such that $mes(E \cap (\Omega_\delta \setminus \Omega_\epsilon)) > 0$. We have
Ω_ϵ for m big enough, and therefore

$$\int_{\Omega_\delta} (y_m - y^*)^2 dx \geq \int_{E \cap (\Omega_\delta \setminus \Omega_\epsilon)} (y_m - y^*)^2 dx = \int_{E \cap (\Omega_\delta \setminus \Omega_\epsilon)} y^{*2} dx > 0 \tag{1.12}$$

he other hand, $y_n \to y^*$ in $L_2(\Omega_\delta)$, $n \to \infty$ follows from the weak convergence of (y_n)
in V_δ and Rellich's Theorem. Thus we arrive at a contradiction with (1.12), i.e.,
0 in $\Omega_\delta \setminus \Omega$. Denoting $y = y^* |_\Omega$, we obtain $y = 0$ on Γ, i.e., $y \in V(v)$. Let a $z \in V(v)$
ven. There exists a sequence (ϕ_n), $\phi_n \in C^\infty(\Omega)$ such that supp $\phi_n \cap \Gamma = \emptyset$, $\phi_n \to z$
(Ω). (Or in $H^1(\Omega_\delta$ if the extensions of Z and ϕ_n are considered). We may write

$$\nabla y_m, \nabla \phi_n)_{0,\Omega_m} + \alpha_0 (y_m, \phi_n)_{0,\Gamma_{N_m}} = (f, \phi_n) 0, \Omega_m$$

For all m sufficiently big. Passing to the limit with $m \to \infty$ we obtain

$$(\nabla y^*, \nabla \phi_n)_{0,\Omega_\delta} + \alpha_0 (y^*, \phi_n)_{0,\Gamma_{N_\delta}} = (f, \phi_n)_{0,\Omega_\delta} \quad (\Gamma_\delta = \partial\Omega_\delta \setminus \Gamma_\delta)$$

Then passing to the limit with $n \to \infty$ we are led to

$$(\nabla y^*, \nabla z)_{0,\Omega_\delta} + \alpha_0 (y^*, z)_{0,\Gamma_{N_\delta}} = (f, z)_{0,\Omega_\delta}$$

As $z = 0$, $y^* = 0$ in $\Omega_\delta \backslash \Omega$, we may replace in (1.13) Ω_δ by Ω and Γ_δ by Γ. Conseq
it holds

$$y^* = y^*(v)$$

The infinition of y_m implies

$$\|\nabla y_m\|^2_{0,\Omega_m} + \alpha_0 \|y_m\|^2_{0,\Gamma_{N_m}} = (f, y_m)_{0,\Omega_m}$$

Using the extensions and the weak convergence in V_δ , we obtain

$$\|\nabla y_m\|^2_{0,\Omega_\delta} + \alpha_0 \|y_m\|^2_{0,\Gamma_{N_\delta}} = (f, y_m)_{0,\Omega_\delta} \to (f, y^*)_{0,\Omega_\delta}$$
$$= (f, y^*)_{0,\Omega} = \|\nabla y^*\|^2_{0,\Omega_\delta} + \alpha_0 \|y^*\|_{0,\Gamma_{N_\delta}}$$

Since the trace of operator is completely continuous and $y_m - y^*$ (weakly), in V_δ, v
$\gamma y_m - \gamma Y^*$ in $L_2(\Gamma_{N_\delta})$, and

$$\|y_m\|^2_{0,\Gamma_{N_\delta}} \to \|y^*\|^2_{0,\Gamma_{N_\delta}}$$

Combining (1.15) with (1.16) we arrive at

$$|y_m|^2_{1,\Omega_\delta} \to \|y^*\|^2_{1,\Omega_\delta}$$

Using also the Rellich's theorem we obtain

$$\|y_m\|^2_{1,\Omega_\delta} \to \|y^*\|^2_{1,\Omega_\delta}$$

Then the strong convergence $y_n \to y^*$ in V_δ follows from (1.18) and the weak
gence.

We consider the case $i = 1$. From (1.18) and (1.14) we deduce

$$\lim_{m\to\infty} J_1(w_m) = \lim_{m\to\infty} J_1(y_m) = \lim_{m\to\infty} [\|y_m - z_0\|^2_{0,\Omega_\delta} - \|z_0\|^2_{0,\Omega_\delta\setminus\Omega_m}]$$
$$= \|y^* - z_0\|^2_{0,\Omega_\delta} - \|z_0\|^2_{0,\Omega_\delta\setminus\Omega} = \|y(v) - z_0\|^2_{0,\Omega} = J_1(y(v)) = J_1(v) = \inf_{w\in U_{ad}} J_1(w)$$

Hence v is a solution

For the case $i = 2$ we have

$$\lim_{m\to\infty} J_2(w_m) = \lim_{m\to\infty} J_2(y_m) = \lim_{m\to\infty} \|\nabla y_m\|^2_{0,\Omega_m} = \|\nabla y^*\|_{0,\Omega_\delta}$$
$$= \|\nabla y(v)\|^2_{0,\Omega} = J_2(y(v)) = J_2(v) = \inf_{w\in U_{ad}} J_2(w)$$

Thus v is the solution. Q.E.D.

2. DUAL FORMULATION OF THE STATE PROBLEM

It is clear that the state problem (1.8) is equivalent to the following one $y(w) \in V(w)$ such that

$$L(y(w)) \le L(z(w)) \quad \forall z(w) \in V(w)$$
$$\text{where} \quad L(z(w)) = a(z,z)/2 + A(z,z)/2 - (f,z)_{0\Omega(w)}$$

ion 1. Let Γ be the boundary of the domain $\Omega(w)$. Let $v \in H^{1/2}(\Gamma)$, $v = 0$ on
note $Zv = \bar{v} \in V(w)$ an arbitrary extension of v into $\Omega(w)$ (see e.g. [3], p. 103 for
ension). Let $M_f(w) \subset [L_2(\Omega(w))]^2$ be the set of vector functions λ such that the
or $\Gamma(\lambda)$ defined through the relation

$$< G(\lambda), v >= \int_{\Omega(w)} (\lambda \text{grad} \bar{v} - f\bar{v}) dx,$$

$M_f(w)$ into $H^{-1/2}(\Gamma_{Nw})$.. (Here $H^{-1/2}(\Gamma_{Nw})$ is the space of linear continuous func-
n $H^{1/2}(\Gamma_{Nw}) \equiv \gamma.V(w)$).

rk 2.1. From the definition of $G(\lambda)$ it follows that

$$| < G(\lambda), v > | \leq C(\lambda) \|v\|_{1/2, \Gamma Nw}$$

e values of $G(\lambda)$ do not depend on the extension Z. Moreover, $\lambda \in M_f$ satisfies the
ng condition (in the sense of distributions)

$$div \lambda + f = 0 \quad \text{in} \quad \Omega(w)$$

tion 2. Let $\Lambda_f(w) \subset M_f(w)$ be the set of all $\lambda \in M_f(w)$ such that $G(\lambda) \in L_2(\Gamma_{Nw})$.
will be called the set of admissible functions. Denote

$$M_w = \bigcup_{f \in L_2(\Omega(w))} \Lambda_f(w) \tag{2.2}$$

the definition of M_w it is readily seen that for every $\lambda \in M_w$ there exists $f(\lambda)$ (where
$iv\lambda$) such that

$$G(\lambda) \in L_2(\Gamma_{Nw})$$

et M_w is a linear manifold and G is linear on M_w. We define a bilinear form on
M_w as follows:

$$(\lambda, \mu)_{H(w)} = \int_{\Omega(w)} \sum_{i=1}^{2} \lambda_i \mu_i \, dx + \alpha_0^{-1} \int_{\Gamma_{Nw}} G(\lambda) G\mu) dS \tag{2.3}$$

is clear that there exists constants $C_3, C_4 > 0$ such that

$$C_3 \|\lambda\|^2_{Mw} \leq \|\lambda\|^2_{H(w)} \equiv (\lambda, \lambda)_{H(w)} \leq C_4 \|\lambda\|_{Mw} \tag{2.4}$$

$$\text{where} \|\lambda\|^2_{Mw} = \sum_{i=1}^{2} \|\lambda_i\|^2_{0,\Omega(w)} + \|G(\lambda)\|^2_{0,\Gamma_{Nw}},$$

$\min(1, \alpha_0^{-1})$, $C_4 = \max(1, \alpha_0^{-1})$. The bilinear form (2.3) is symetric, continuous and
ely definite (with the norm $\| \cdot \|_{Mw}$), hence it represents a scalar product. The
old M with the scalar product (4.3) will be denoted by $H(w)$. Let us define

$$H_1 = \{\lambda \mid \lambda \in Mw, \ \exists v \in V(w), \lambda = \nabla v, \alpha_0 + G(\lambda) = 0 \ \text{on} \quad \Gamma_{Nw}\} \quad H_2 = \Lambda_0(w)$$

forth we write $\lambda = \lambda(v)$ if $\lambda = \nabla v$

a 2.1. H_1 and H_2 orthogonal subsets of $H(w)$.

roof: Let $\lambda \in H_1$, $\mu \in H_2$. we then have

$$(\lambda, \mu)_{H(w)} = \int_{\Omega(w)} \mu.\text{grad} v dx - \int_{\Gamma_{Nw}} v.G(\mu).ds = 0$$

making use of the definition of Λ_0, i.e., $H_1 \perp H_2$ Q.E.D.

Theorem 2. (Principle of Minimum Complementary Energy). Let $y(w)$ be the of the problem (2.1). Then the objective functional

$$S(\lambda) = 1/2\|\lambda\|^2_{H(w)}$$

attains its minimum on the set $\Lambda_f(w)$ of admissible functions, iff

$$\lambda = \lambda(y(w)) = \nabla y(w)$$

Moreover, it holds

$$S(\lambda(y(w))) + L(y(w)) = 0$$

Proof: For brevity we shall omit w what follows. First, we shall show that $\lambda($ In fact, y is the solution of (4.1), hence it holds

$$\int_\Omega \lambda(y)\text{grad } v dx - \int_{\Gamma_N} \alpha_0 yv\, ds = (f, v)_{0,\Omega} \quad \forall v \in V(w).$$

Consequently, we arrive at

$$\int_\Omega [\lambda(y)\text{grad } v - fv]dx =< G(\lambda(y)), v >= -\int_{\Gamma_N} \alpha_0 yv ds,$$

i.e., $G(\lambda(y)) = \alpha_0 y \in L_2(\Gamma_N)$, hence $\lambda(y) \in \Lambda_f \cap H_1$

Let λ arbitrary element of Λ_f. From the definition 1 we have: for all $v \in H^{1/2}($I 0 on Γ_N, $\overline{w} = Zv$

$$< G(\lambda), v > = \int_\Omega (\lambda \text{grad} \bar{w} - f\bar{w})dx,$$
$$< G(\lambda(y)), v > = \int_\Omega (\lambda(y)\text{grad}\bar{w} - f\bar{w})dx,$$

therefore we obtain

$$< G(\lambda - \lambda(y)), v >= \int_\Omega [\lambda - \lambda(y)]\text{grad}\bar{w}dx,$$

making use of the linearity of $G(\lambda)$ on H, i.e., $\lambda - \lambda(y) \in \Lambda_0 \equiv H_2$ for any $\lambda \in \Lambda_f$.

We may write (on the basis of Lemma 2.1).

$$S(\lambda) = 1/2\|\lambda\|^2_H = 1/2\|\lambda - \lambda(y)\|^2_H + 1/2\|\lambda\|^2_H$$

From this it is clear that $S(\lambda)$ attains its minimum on Λ_f, iff $\lambda = \lambda(y) = \nabla y$

We have

$$2S(\lambda(y)) = \|\lambda(y)\|^2_H = \sum_{i=1}^{2} \|\lambda_i(y)\|^2_{0,\Omega} + \alpha_0^{-1} \int_{\Gamma_N} G(\lambda(y))^2 ds$$
$$= -\sum_{i=1}^{2} \|\lambda_i(y)\|^2_{0,\Omega} - 2\int_\Omega y\nabla y dx - \alpha_0 \int_{\Gamma_N} y^2\, ds$$
$$= -\sum_{i=1}^{2} \|dy/dx_i\|^2_{0,\Omega} - \alpha_0 \int_{\Gamma_N} y^2\, ds - \int_\Omega fy\, ds = 2L(y)$$

$\lambda(y)) + L(y) = 0$. Q.E.D.

e now may rewrite the cost functional $J_2(w)$ as follows

$$J_2(w) = \|\lambda(y(w))\|^2_{0,\Omega(w)} = J_2^*(\lambda(w)) \tag{2.5}$$

e optimization problem has the followingequivalent form

$$J_2^*(\lambda(v)) = \min_{w \in U_{ad}} J_2^*(\lambda(w)) \tag{2.6}$$

$S\lambda(w)$ is the solution of the optimization problem

$$S(\lambda(w)) \leq S(\lambda) \qquad \forall \lambda \in \Lambda_f(w) \tag{2.7}$$

be a solution of the optimization problem (1.2) for $i = 2$. Then

$$J_2^*(\lambda(v)) = J_2(v) \leq J_2(w) = J_2^*(\lambda(w)) \qquad \forall w \in U_{ad}$$

s from (2.5). Hence v is also a solution of the equivalent optimization problem Using Theorem 1.1 we conclude that the problem (2.6) has at least one e solution.

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VỀ SỰ TỒN TẠI NGHIỆM VÀ PHÁT BIỂU BIẾN PHÂN ĐỐI NGẪU CHO BÀI TOÁN TỐI ƯU MIỀN ĐỐI VỚI PHƯƠNG TRÌNH ELIPTIC

Trần Văn Bốn

Học viện Hải quân Nha Trang

ới mục đích phân tích phần tử hữu hạn trực tiếp và đối ngẫu cho bài toán tối ưu đối với bài toán Eliptic, trong bài viết này chúng ta chứng minh sự tồn tại nghiệm ài toán trực tiếp mà chủ yếu là sự tồn tại phần tử cực tiểu của các phiếm hàm giá ập hợp các hàm chấp nhận được và trên cơ sở nguyên lý năng lượng phát biểu phân đối ngẫu của bài toán đã được trình bày, mối liên hệ giữa lời giải yếu của án trực tiếp và lời giải đối ngẫu đã được chứng minh.