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# ON THE COMPARISON PROBLEM OF THE STABILI FOR NON LINEAR DYNAMICAL SYSTEMS PERTURBED BY SMALL NOISE

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ABSTRACT

*This paper deals with the comparison problem of stability of differential equ perturbed by non - linear small noise. We suppose that the the linear system*

$$dZ_t = a(t,\omega)Z_t\,dt + A(t,\omega)Z_t\,dW_t\,; \qquad Z_0 = z \in R^d$$

*is strictly stabler than the system*

$$dY_t = b(t,\omega,Y_t)\,dt + B(t,Y_t,\omega)\,dW_t\,; \qquad Y_0 = y \in R^d$$

*then, under the assumption of the regulity of (1), it is proved that the system*

$$dX_t = \big(a(t,\omega)X_t + f(t,X_t)\big)\,dt + A(t,\omega)Z_t\,dW_t\,; \qquad Z_0 = z \in R^d$$

*is strictly still stabler than System (2) provided $f(t,x)$ satisfies the condition*

$$|f(t,x)| \le k.\min\{|x|^\alpha, |x|^{1-\beta}\}\,; \quad \alpha > 1 > \beta > 0$$

## I. INTRODUCTION

As is known, investigating of the fact whether a given dynamical system is st unstable is important in both theory and application. Therefore, many definitions stability of systems are given (see, for example, [6], [7],[3]) and there are a vast a of works dealing with criteria by which we know whenever a given differential equa stable ( see [6], [7], [3],...). Among these criteria, the Lyapunov exponents of soluti a powerful tool mainly because of its importance for explaining chaotic behaviour systems (see [1], [2],...). Furthermore, in order to study the stability of linear syst general, we have only to consider their Lyapunov exponents. If they are negativ their trivial solution $X = 0$ must be stable.

But as to our knowledge, there is no definition which allows us to compa "degree" of the development of systems even they are defined in a same space an the same dimension. In some cases, this comparison is necessary because many te problems require us to choose a system which is the less chaotic the better amo given systems.

On the other hand, studying the Lyapunov exponent of a function means t compare this function with exponential functions. However, the class of expo

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s contains not many informations of growth rates because they are monotonous. re, if we replace this class by a larger one, we hope to have more informations he behaviour of the considered function.

ing on this idea we give a concept for comparing the growth rate of two systems. ssical definition of stabilities can be obtained by comparing the considered system e trivial system $X \equiv 0$.

ides, by Lyapunov Theorem for the Stability (see [4],pp. 267), if the linear system entialy stable then it is still stable under small noise. We want here to generalise ult in the point of view of preserving the "order" of stabilities. It is proved that m (1) is stabler than (2), then it is stabler than (2) under small non linear noise. e article is organized as follows: Section II introduces a definition for comparing bility between two systems whose states are described by stochastic equations in e of real noise or white noise and we give some remarks on this definition. In III, we formulate the main result. It is shown that under the small noise $f(t,x)$ regulity of the linear system, System (3) is stabler than (1).

## II. COMPARISON OF GROWTH RATE OF DYNAMICAL SYSTEMS

$(\Omega, \mathcal{F}_t, t \geq 0, P)$ be a stochastic basis satisfying the standard conditions (see [5]) $_t, t \geq 0)$ be a $d$ - dimension wiener process defined on $(\Omega, \mathcal{F}_t, t \geq 0, P)$. We consider astic system described by the following equation

$$\begin{cases} dX_t &= a(t, X_t, \omega)\, dt + A(t, X_t, \omega) dW_t \\ X_0 &= x \in R^d \end{cases} \tag{2.1}$$

or all $x \in R^d$ , $((a(t,x))$ and $(A(t,x))$ are two stochastic processes $\mathcal{F}_t$- adapted with n $R^d$ and in the space of $d \times d$ - matrices respectively such that

$$a(t,0) \equiv 0 \qquad A(t,0) \equiv 0 \qquad P - a.s \tag{2.2}$$

ppose that for any $x \in R^d$, Equation (2.1) has a unique strong solution. Let us he classical definition of stability in Lyapunov's sense. Denote by $X(t,x,\omega)$ the n of (2.1) starting from $x$ at $t = 0$. From (2.2), it follows that $X \equiv 0$ is a solution ation (2.1).

**ion 2.1.** *The trivial solution $X \equiv 0$ is said to be stable if for any $\epsilon > 0$*

$$\lim_{x \to 0} P\left( \sup_{0 \leq t < \infty} |X(t,x,\omega)| > \epsilon \right) = 0 \tag{2.3}$$

5] pp. 206 ). It is known that in fact considering whether a system is stable means we compare its solutions with constant functions because the relation $X(t,x,\omega)| < \epsilon$ means $|X(t,x,\omega)| \leq \epsilon(t)$ for any $t > 0$ where $\epsilon(t) = \epsilon \ \forall\, t > 0$ .

gret that this definition gives no information when the solution $X(t,x)$ tends to e or to 0. Thus it requires us to consider a larger class of functions to know more avior of systems. We now realise this idea.

side of Equation (2.1) we consider the equation

$$\begin{cases} dY_t &= b(t, Y_t, \omega)\, dt + B(t, Y_t, \omega) dW_t \\ Y_0 &= y_0 \in R^n \end{cases} \tag{2.4}$$

$(b(t,y))$ and $(B(t,y))$ satisfy the same hypothesis as $(a(t,y))$ and $(A(t,y))$, i.e.,

$$b(t,0) \equiv 0 \qquad B(t,0) \equiv 0 \qquad \forall t \geq 0 \qquad P - a.s \tag{2.5}$$

We write for $Y(t,x)$ the solution of (2.4) starting from $y$ at $t=0$

Let $C$ the set of all positive continuous functions from $[0,\infty)$ into $R^+$ and subset of $C$.

**Definition 2.2.** *The trivial solution $X \equiv 0$ of System (2.1) is said to be stabler t solution $Y \equiv 0$ of System (2.4) in the comparing class $\mathcal{M}$ if for any $q \in M$, the rel*

$$\lim_{y \to 0} P\{|Y(t,y)| \le q_t \quad \text{for all} \;.\; t \ge 0\} = 1$$

*follows that*

$$\lim_{x \to 0} P\{|X(t,x)| \le q_t \quad \text{for all} \quad t \ge 0\} = 1$$

Definition 2.2 is an extension of the classical one of stability. Indeed, we h following theorem.

**Theorem 2.3.** *System (2.1) is stable in sense of (2.3) if it is stabler than the trivial*

$$\dot{Y} = 0, \qquad Y_0 = y \in R^d$$

*on the class $C$.*

*Proof*: If (2.1) is stabler than (2.8), then it is easy to see that (2.1) is stab every solution of (2.8) is constant. Inversselly, suppose that (2.1) is stable and $q$ $\inf_{0 \le t < \infty} q_t = 0$ then Equality (2.6) does not hold. Meanwhile if $\inf_{0 \le t < \infty} q_t = k > 0$ th hods which implies that

$$1 = \lim_{x \to 0} P(\sup_{0 < t < \infty} |X(t,x)| \le \alpha) \le \lim_{x \to 0} P(|X(t,x)| \le q_t \quad t \ge 0)$$

i.e System (2.1) is stabler than System (2.4). Moreover, it is easy to prove that

**Theorem 2.2:** *If $\mathcal{M} \subset C$ consists of all functions having the exact limit as $t \to$ every stable system is stabler than any unstable system.*

**Example:** Both two systems

$$\ddot{X} - \dot{X} + 2X = 0$$
$$\ddot{Y} - 2\dot{Y} + 2Y = 0$$

are unstable. But it is easy to see that (A) is stabler than (B) in $C$.

## III. LINEAR REGULAR SYSTEM.

We introduce the so-called regular system as in [4]. Let us consider the lineai

$$dZ_t = A_t Z_t \, dt + B_t Z_t \, dW_t \qquad Z_0 = z \in R^d$$

where $A_t$, $B_t$ are two stochastic processes with values in $d \times d$- matrices satisf condition.

$$P\{\int_0^T |A_t| dt < \infty\} = P\{\int_0^T |B_t| dt < \infty\} = 1; \quad \text{for any} \quad T > 0$$

This condition ensures the existence of strong solutions of (3.1).

$Z(t,z)$ be the solution of (3.1) starting from $z$. We write for $\lambda[z]$ the Lyapunov nt of $Z(t,z)$ defined by

$$\lambda[z] = \limsup_{t\to\infty} \frac{1}{t} \ln |Z(t,z)| \tag{3.2}$$

case where the limit in (3.2) exists, we say that $Z(t,z)$ has an exact exponent. is known that (see [1], [6]...) the Lyapunov spectrum of the solution of (3.1) s of $n$ random variables, namely,

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \tag{3.3}$$

tion 2.3. *(See [4] pp. 165). System (3.1) is said to be regular if there exists a nental system of solutions $Z(t)$ such that the column vectors of $Z(t)$ has the exact ent and takes all values $\lambda_i$, $i = 1, 2, \ldots, d$ in (3.3)*

t the comparing class $\mathcal{M}$ consist of elements $q \in C$ having the exact limit

$$\lim_{t\to\infty} \frac{1}{t} \ln q_t = \overline{q} \tag{3.4}$$

y that (2.1) is strictly stabler than (2.4) if the condition (2.7) is replaced by: There $\mathcal{M}$ such that $\overline{q^*} < \overline{q}$ and

$$\lim_{y\to 0} P\{|X(t,x)| \le q_t^* \qquad \forall t \ge 0\} = 1 \tag{2.7'}$$

asy to see that (2.1) is strictly stabler than (2.8) in $\mathcal{M}$ if and only if (2.1) is ently stable.

em 3.2. *Suppose that (3.1) is regular and strictly stabler than the system*

$$dY_t = a(t, Y_t)\, dt + \sigma(t, Y_t)\, dW_t \qquad Y_0 = y \in R^d \tag{3.5}$$

*the perturbed system*

$$dX_t = [A_t X_t + f(t, X_t)]\, dt + B_t X_t\, dW_t \qquad X_0 = x \in R^d \tag{3.6}$$

*strictly stabler than (3.5). Where $f(t,x)$ is a locally Lipchitz function satysfying ndition: There are constants $\alpha > 1 > \beta > 0$; $K > 0$ such that*

$$|f(t,x)| \le K.\min(|x|^\alpha, |x|^{1-\beta}) \tag{3.7}$$

*roof:* From the assumption of the regulity of (3.1), we can find a fundamental n of solutions of (3.1), namely $Z(t)$, such that: if

$$\Phi(t) = Z(t).\exp\{-\Lambda t\}, \qquad \Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$$

$$\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(t)| = \lim_{t\to\infty} \frac{1}{t} \ln |\Phi^{-1}(t)| = 0 \tag{3.8}$$

efore, for any $\gamma > 0$ there is a random variable N such that

$$|Z(t).Z^{-1}(s)| \le N.\exp\{(\lambda_d + \gamma)t - (\lambda_d - \gamma)s\} \qquad \text{P- a.s} \tag{3.9}$$

Let $q \in \mathcal{M}$ such that

$$\lim_{y \to 0} P\{|Y(t,y)| \leq q_t \qquad \forall t \geq 0\} = 1$$

Since (3.1) is strictly stabler than (3.5), then there exists $q^* \in \mathcal{M}$, $\overline{q}^* < \overline{q}$ anl

$$\lim_{z \to 0} P\{|Z(t)z| \leq q_t^* \qquad \forall t \geq 0\} = 1$$

This equality implies $\lambda_d \leq \overline{q}^* < \overline{q}$. Therefore, we can choose $\gamma$ in the inequality (3.9 that

$$(*) \quad \frac{2\gamma}{\beta} < \overline{q} \quad \text{and} \quad \lambda_d + \gamma < \overline{q} \quad \text{when} \quad \lambda_d \geq 0$$

$$(**) \quad \lambda_d + \gamma < \overline{q} \qquad \text{and} \quad (\alpha - 1)\lambda_d + (\alpha + 1)\gamma < 0 \quad \text{when} \quad \lambda_d < 0$$

It is easy to see that (3.16) is equivalent to

$$X_t = Z(t)x + \int_0^t Z(t).Z^{-1}(s)f(s, X_s)ds$$

Therefore

$$|X_t| \leq |Z(t)x| + \int_0^t |Z(t).Z^{-1}(s)f(s, X_s)|ds$$
$$\leq |Z(t)x| + K\int_0^t |Z(t).Z^{-1}(s)| \min(|X_s|^\alpha, |X_s|^{1-\beta})ds$$

We consider two cases:
a). $\lambda_d \geq 0$ By (3.13) and (3.9) we get

$$|X_t| \leq \exp\{(\lambda_d + \gamma)t\} \left[N.|x| + K.N\int_0^t \exp\{-(\lambda_d - \gamma)s\}.|X_s|^{1-\beta}ds\right]$$
$$= \exp\{(\lambda_d + \gamma)t\} \left[N.|x| + K.N\int_0^t \exp\{((2-\beta)\gamma - \beta\lambda_d).s\}.|e^{-(\lambda_d+\gamma).s}X_s|^{1-\beta}d.\right.$$

By virtue of of Bihari' s inequality ( see [4] pp. 110 ) we get

$$|X_t| \leq \exp\{(\lambda_d + \gamma)t\} \left[N.|x|^\beta + \beta K.N\int_0^t \exp\{\sigma s\}\, ds\right]^{\frac{1}{\beta}} \qquad \text{P - a.s}$$

where $\sigma =: (2-\beta)\gamma - \beta\lambda_d$. Hence, there exists a random variable $M$ such that $x: |x| < 1$ we have

$$|X_t| \leq M.\exp\{(\lambda_d + \gamma + \overline{\sigma})t\} \qquad \text{P- a.s} \qquad where \quad \overline{\sigma} = \max(0.\frac{\sigma}{\beta})$$

For any $\epsilon > 0$ fixed, it follows from (3.12 (*)) that there exists a random $T_1 > 0$ su

$$P\{M.\exp\{(\lambda_d + \gamma + \overline{\sigma})t\} < q_t \qquad \forall t \geq T_1\} > 1 - \epsilon/2$$

On the other hand, on $[0, T_1]$, the solutions $X(t,x)$ depend continuously o the condition $x$, then we can choose an $\delta > 0$ such that

$$P\{|X(t,x)| \leq q_t \qquad \forall t \in [0, T_1]\} \geq 1 - \epsilon/2 \qquad when \qquad |x| < \delta$$

g (3.14) and (3.15) it yields

$$P\{|X(t,x)| \le q_t \quad \forall t \le 0\} \ge 1-\epsilon \quad when \quad |x| < \delta$$

ns that (3.6) is stabler than (3.5).
). Using (3.13) we have

$$|X_t| \le |Z(t)x| + +K \int_0^t |Z(t).Z^{-1}(s)|.|X_s|^\alpha ds$$
$$\le e^{(\lambda_d+\gamma)t}[N.|x| + K.N \int_0^t \exp\{(\lambda+\gamma)s - (\lambda_d-\gamma)s\}|X_s|^\alpha ds]$$
$$\le e^{(\lambda_d+\gamma)t}[N.|x| + K.N \int_0^t e^{\sigma s}|e^{-(\lambda_d+\gamma)s}.X_s|^\alpha ds]$$

$=: (\alpha-1)\lambda_d + (\alpha+1)\gamma$
virtue of Bihari's inequality which $\alpha > 1$ (see [4], pp 110), it yields

$$|X(t,x)| \le \frac{N.|x_0|\exp\{(\lambda_d+\gamma)t\}}{\left[1-(\alpha-1)|x_0|^{\alpha-1} \times \int_0^t e^{\sigma s}\, ds\right]^{\frac{1}{\alpha-1}}}$$

$_0$ is small.
ng the same argument as above, we conclution that $\forall\epsilon > 0$, there is an $\delta > 0$ such $|x| < \delta$ then

$$P\{|X(t,x)| \le q_t \quad \forall t \ge 0\} > 1-\epsilon$$

the result follows.

**ry 3.3** *(See [4] pp.267 ). If the top Lyapunov exponent of (3.1) is negative, then turbed system (3.6) is stable.*

**le 3.4** The assumption of regulity of (3.1) is satysfied when $(A_t)$ and $(B_t)$ are tionary processes. The matter of fact is that is that in this case (3.2) generates a e $(Z(t))$ and by Floqué's representation ( see [1] and [2])

$$Z(t) = S(t)\exp\{\Lambda t + o(t)\}, \quad \frac{o(t)}{t} \to 0 \text{ as } t \to \infty$$

$S(t)$ is a random process with values in the sphere $\{z \in R^d : |z| = 1\}$
note that Theorem 3.2 may not be true if conditions 3.7 is violtaed as the following es:
**le 3.5** Let us conside the logistic equation

$$dX_t = (\alpha X_t + \sqrt[3]{X_t})\,dt + \sigma.X_t \circ dW_t \tag{3.15}$$

o denotes the Stranovich equation. It is easy to see that (3.15) has the solutions

$$|X_t| = x.\exp\{\alpha t + \sigma.W_t\} \times \left[|x_0|^{\frac{2}{3}} + \frac{2}{3}\int_0^t \exp\{-\frac{2}{3}(\alpha s + \sigma.W_s)\}ds\right]^{\frac{3}{2}}$$

other hand System (3.15) is a perturbation of

$$dZ_t = \alpha.Z_t + \sigma.Z_t \circ W_t \tag{3.16}$$

which is stabler than

$$dY_t = \frac{\alpha}{2}.Y_t\,dt$$

when $\alpha < 0$. Therefore, (3.16) is stabler than (3.17), but (3.15) is not stablr tha i.e., the assertion of Theorem 3.2 is not true. We remark that both Systen 3.15 are unstable.

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# VỀ BÀI TOÁN SO SÁNH TÍNH ỔN ĐỊNH CỦA HỆ ĐỘNG HỌC CHỊU NHIỄU NHỎ

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Bài báo đưa ra quan niệm mới về sự so sánh tính ổn định của hai hệ đ. Định nghĩa cổ điển về ổn định có thể nhận được bằng cách so sánh hệ là ch tầm thường. Bài báo cũng đề cập tới việc mở rộng định lý Lyapunov về tính theo quan điểm bảo toàn thứ tự ổn định của hệ động học chịu nhiễu phi tuyến