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Weak Laws of Large Numbers of Cesaro Summation for Random Arrays

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Abstract: In this paper, we establish weak laws of large numbers with or without random indices for Cesaro summation for random arrays of random elements in Banach spaces. Our results are more general and stronger than some well-known ones. AMS Subject classification 2000: 60B11, 60B12, 60F05, 60G42.

Keywords: p-uniformly smooth Banach space, double arrays of random elements, double arrays, random indices, weak laws of large numbers.

1. Introduction

Consider a double array $\{X_{mn}; m \geq 1, n \geq 1\}$ of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in a real separable Banach space $\mathbb{E}$ with norm $\|.\|$. Let $\{u_n; n \geq 1\}$ and $\{v_n; n \geq 1\}$ be sequences of positive integers, let $\{T_n; n \geq 1\}$ and $\{\tau_n; n \geq 1\}$ be sequences of positive integer-valued random variables. In the current work, we extend weak laws of large numbers of Cesaro summation for random arrays and for double arrays with random indices.

Limit theorems for weighted sums (with or without random indices) for random variables (realvalued or Banach space-valued) are studied by many authors (see, e.g., Wei and Taylor [1], OrdonezCabrera [2], Adler et al. [3], Sung et al. [4]). Recently, Dung [5] obtained the weak law of largenumbers with random indices for double arrays of random elements. In this paper, we establish theweak laws of large numbers with or without random indices for Cesaro summation for random arrays ofrandom elements in a p- uniformly smooth Banach space.

2. Preliminaries

For $\alpha > -1$, we let

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!}, \text{n = 1,2,3,... and } A_0^\alpha = 1.$$

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Then the definition of Cesaro summability for array is extended as follows.

Definition 1. Let $\alpha, \beta > 0$. The array $\{x_{mn}; m,n \geq 0\}$ is said to be (C, α, β) -summable iff

$\frac{1}{A_m^{\alpha} A_n^{\beta}} \sum_{k,l=0}^{m,n} A_{n-k}^{\alpha-1} A_{n-l}^{\beta-1} x_{kl}$ converges as $m, n \to \infty$.

It is easy to see that (C, 1, 1)-convergence is the same as convergence of array of the arithmetic mean.

Here we collect some facts that will be used on and off in general without specific reference.

Firstly, we have

$$A_n^{\alpha} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)} \text{ as } n \to \infty \tag{1.1}$$

Secondly, we use the fact that if $\{a_k, k \geq 1\}$ is a sequence of numbers such that $A_n \nearrow \infty$ as $n \to \infty$ where $A_n = \sum_{k=1}^{n} a_k$ and $x_n \to 0$ as $n \to \infty$ then

$$\frac{1}{A_n} \sum_{k=1}^{n} a_k x_k \to 0 \text{ as } n \to \infty. \tag{1.2}$$

For $a, b \in \mathbb{R}$, min{a, b} and max{a, b} will be denoted, respectively, by $a \wedge b, a \vee b.$ Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

Technical definitions that are relevant to the current work will be discussed in this section. Scalora [6] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element X and sub σ–algebra G of $\mathcal{F}$, the conditional expectation $E(X \mid G)$ is defined analogously to that in the random variable case and enjoys similar properties.

A real separable Banach space $\mathbb{E}$ is said to be *p-uniformly smooth* $(1 \leq p \leq 2)$ if there exists a finite positive constant C such that for all martingales $\{S_n; n \geq 1\}$ with values in $\mathbb{E}$,

$$\sup_{n \geq 1} E \| S_n \|^p \leq C \sum_{n=1}^{\infty} E \| S_n - S_{n-1} \|^p .$$

It can be shown by using classical methods from martingale theory that if $\mathbb{E}$ is p-uniformly smooth, then for each $1 \leq r < \infty$ there exists a finite constant C such that

$$E \sup_{n \geq 1} \| S_n \|^r \leq CE \left(\sum_{n=1}^{\infty} \| S_n - S_{n-1} \|^p \right)^{\frac{r}{p}} .$$

Clearly, every real separable Banach space is 1-uniformly smooth and the real line (the same as any Hilbert space) is 2-uniformly smooth.

It follows from the Hoffmann-Jøgensen and Pisier [7] that if a Banach space is p-uniformly smooth, then it is of Rademacher type p. But the notion of p-uniformly smooth is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness.

Let $\mathcal{F}_{kl}$ be the σ-field generated by the family of random elements $\{X_{ij}; i < k \text{ or } j < l\}$, $\mathcal{F}_{1,1}$ = $\{\varnothing;\Omega\}$. The following lemma which is due to Dung [5] establishes a maximal inequality for double sums of random elements in martingale type p Banach spaces.

Lemma 2. Let $1 < p \le 2$. Let $\{X_{ij}; 1 \le i \le m, 1 \le j \le n\}$ be a collection of mn random elements in a real separable Banach space such that $E(X_{ij} \mid \mathcal{F}_{kl}) = 0$ for all $1 \le i \le m, 1 \le j \le n.$ Then,

$$E \max_{\substack{1\le k\le m\\ 1\le l\le n}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l} X_{ij} \right\|^p \le C \sum_{i=1}^{m}\sum_{j=1}^{n} E\| X_{ij} \|^p \tag{1.3}$$

where the constant C is independent of m and n.

Random elements $\{X_{mn}; m \ge 1, n \ge 1\}$ are said to be *stochastically dominated* by a random element X if for some finite constant D

$P\{\| X_{mn} \| > t\} \le DP\{\| DX \| > t\},\ t \ge 0, m \ge 1, n \ge 1.$

3. The main results

Let $\{X_{mn}; m \ge 1, n \ge 1\}$ be an array of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space $\mathbb{E}$ with norm $\|.\|$, $\mathcal{F}_{kl}$ be a σ-field generated by $\{X_{ij}; i < k \text{ or } j < l\}$, $\mathcal{F}_{1,1}$ = $\{\varnothing;\Omega\}$. Let $\{u_n; n \ge 1\}$, $\{v_n; n \ge 1\}$ be sequences of positive integers such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$. For any set A, we denote I (A) the indicator function, i.e,

$$I(A)(\omega) = \begin{cases} 1 & if\ \omega \in A \\ 0 & if\ \omega \notin A \end{cases}$$

Set

$$Y_{kl}^{mn} = A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} X_{kl} I(A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} \| X_{kl} \| \le A_m^{\alpha} A_n^{\beta})$$

where $\alpha, \beta > 0$

Theorem 3. Let $1 \le p \le 2$, $\alpha, \beta > 0$ and $\mathbb{E}$ be a p-uniformly smooth Banach space. Suppose that

$$\sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P\{A_{m-i}^{\alpha-1} A_{n-j}^{\beta-1} \| X_{ij} \| > A_m^{\alpha} A_n^{\beta}\} \to 0 \text{ as } m \vee n \to \infty \tag{2.1}$$

and

$$\frac{1}{(A_m^{\alpha} A_n^{\beta})^p} \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} E\| Y_{ij}^{mn} - E(Y_{ij}^{mn} \mid \mathcal{G}_{ij}) \|^p \to 0 \text{ as } m \vee n \to \infty. \tag{2.2}$$

Then

$$\max_{\substack{1\le k\le u_m\\ 1\le l\le v_n}} \frac{1}{A_m^{\alpha} A_n^{\beta}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l} A_{m-i}^{\alpha-1} A_{n-j}^{\beta-1} X_{ij} - E(Y_{ij}^{mn} \mid \mathcal{G}_{ij}) \right\| \xrightarrow{p} 0 \text{ as } m \vee n \to \infty. \tag{2.3}$$

Proof. For an arbitrary $\epsilon > 0$,

$$P\left\{\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}X_{ij} - E(Y_{ij}^{mn} \mid G_{ij})\right\| > \epsilon\right\}$$

$$\le P\left\{\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}X_{ij} - Y_{ij}^{mn}\right\| > \epsilon/2\right\}$$

$$+P\left\{\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} (Y_{ij}^{mn} - E(Y_{ij}^{mn} \mid \mathrm{G}_{ij}))\right\| > \epsilon/2\right\}$$

$$\le P\left\{\bigcup_{i=1}^{u_m}\bigcup_{j=1}^{v_n}(A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}\| X_{ij}\| > A_m^\alpha A_n^\beta)\right\}$$

$$+P\left\{\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} (Y_{ij}^{mn} - E(Y_{ij}^{mn} \mid G_{ij}))\right\| > \epsilon/2\right\}$$

$$\le \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P(A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}\| X_{ij}\| > A_m^\alpha A_n^\beta)$$

$$+\frac{2^p}{\epsilon^p (A_m^\alpha A_n^\beta)^p} E\max_{\substack{1\le k\le u_m\\1\le l\le v_n}}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} (Y_{ij}^{mn} - E(Y_{ij}^{mn} \mid \mathrm{G}_{ij}))\right\|^p \text{ (by Markov's inequality)}$$

$$\le \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P(A_{m-k}^{\alpha-1}A_{n-l}^{\beta-1}\| X_{ij}\| > A_m^\alpha A_n^\beta)$$

$$+\frac{C}{(A_m^\alpha A_n^\beta)^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n} E\| Y_{ij}^{mn} - E(Y_{ij}^{mn} \| \mathrm{G}_{ij})\|^p \text{ (by Lemma 2)}$$

$\to 0$ as $m \vee n \to \infty$ (by (3.1) and (3.2)).

The proof is completed. □

Corollary 4. Let $1 \le p \le 2$, $\alpha > 0$, $\beta > 0$ and $\mathbb{E}$ be a p-uniformly smooth Banach space. If

$$\sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P\{A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}\| X_{ij}\| > A_m^\alpha A_n^\beta\} \to \infty \text{ as } m \vee n \to \infty,$$

$$\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}E(Y_{ij}^{mn} \mid \mathrm{G}_{ij})\right\| \xrightarrow{p} 0 \text{ as } m \vee n \to \infty$$

and

$$\frac{1}{(A_m^\alpha A_n^\beta)^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\| E(Y_{ij}^{mn} \mid \mathcal{G}_{kl})\|^p \to 0 \quad \text{as } m\vee n\to\infty, \tag{2.4}$$

then

$$\max_{\substack{1\le k\le u_m\\ 1\le l\le v_n}}\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}X_{ij}\right\| \xrightarrow{p} 0 \quad \text{as } m\vee n\to\infty. \tag{2.5}$$

Remark 5. If the condition (3.4) is replaced by the condition that

$$\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}E(Y_{ij}^{mn}\mid \mathcal{G}_{ij})\right\| \xrightarrow{p} 0 \text{ as } m\vee n\to\infty,$$

then the conclusion (3.5) will be replaced by

$$\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}X_{ij}\right\| \xrightarrow{p} 0 \text{ as } m\vee n\to\infty.$$

The following result is a random index version of Theorem 3.

Theorem 6. Let $1\le p\le 2$, $\alpha,\beta>0$ and $\mathbb{E}$ be a p-uniformly smooth Banach space. Suppose that $\{T_n; n\ge 1\}$ and $\{\tau_n; n\ge 1\}$ are sequences of positive integer-valued random variables such that

$$\lim_{n\to\infty}P\{T_n>u_n\}=\lim_{n\to\infty}P\{\tau_n>v_n\}=0. \tag{2.6}$$

If

$$\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}P\{A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}\|X_{ij}\|>A_m^\alpha A_n^\beta\}\to\infty \text{ as } m\vee n\to\infty$$

and

$$\frac{1}{(A_m^\alpha A_n^\beta)^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}E\|Y_{ij}^{mn}-E(Y_{ij}^{mn}\mid \mathcal{G}_{ij})\|^p\to 0 \text{ as } m\vee n\to\infty,$$

then

$$\max_{\substack{1\le k\le T_m\\ 1\le l\le \tau_n}}\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}(X_{ij}-E(Y_{ij}^{mn}\mid \mathcal{G}_{ij}))\right\| \xrightarrow{P} 0 \text{ as } m\vee n\to\infty.$$

Proof. For arbitrary $\epsilon>0$,

$$P\left\{\max_{\substack{1\le k\le T_m\\1\le l\le\tau_n}}\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}(X_{ij}-E(Y_{ij}^{mn}\mid \mathcal{G}_{ij}))\right\|>\epsilon\right\}$$

$$\le P\left\{\left(\max_{\substack{1\le k\le T_m\\1\le l\le\tau_n}}\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}(X_{ij}-E(Y_{ij}^{mn}\mid \mathcal{G}_{ij}))\right\|>\epsilon\right)\cap\left(T_m\le u_m\right)\cap\left(\tau_n\le v_n\right)\right\}$$

$$+P(T_m>u_m)+P(\tau_n>v_n)$$

$$\le P\left\{\left(\max_{\substack{1\le k\le u_m\\1\le l\le v_n}}\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}(X_{ij}-E(Y_{ij}^{mn}\mid \mathcal{G}_{ij}))\right\|>\epsilon\right)\right\}$$

$$+P(T_m>u_m)+P(\tau_n>v_n)$$

$\to 0$ as $m\vee n\to\infty$, (by (3.6) and Theorem 3)

which completes the proof. □

We shall now prove the following extension of the well-known Feller theorem for Cesaro summation for random arrays of random elements in Banach spaces.

Theorem 7. Let $1\le p\le 2$, $0<\alpha<\beta\le 1$, $\mathbb{E}$ be a p-uniformly smooth Banach space. Suppose that $\{X_{mn};m\ge 1,n\ge 1\}$ is stochastically dominated by a random element X. If

$$\lim_{n\to\infty} nP\{\|X\|>n\}=0, \tag{2.7}$$

then

$$\max_{\substack{1\le k\le m\\1\le l\le n}}\frac{1}{A_m^\alpha A_n^\beta}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}(X_{ij}-E(Y_{ij}^{mn}\mid \mathcal{G}_{ij}))\right\|\xrightarrow{P}0 \text{ as } m\vee n\to\infty.$$

Proof. We verify (3.1) and (3.2) with $u_m=m$, $v_n=n$. For (3.1), we have

$$\sum_{i,j=1}^{m,n}P\{A_{m-i}^{\alpha-1}A_{n-j}^{\beta-1}\|X_{ij}\|>A_m^\alpha A_n^\beta\}\le C\sum_{i,j=1}^{m,n}P\{A_{m-i}^{\alpha-1}A_{n-j}^{\beta-j}\|X\|>A_m^\alpha A_n^\beta\}$$

$$\le C\sum_{i,j=1}^{m,n}P\{i^{\alpha-1}j^{\beta-1}\|X_{ij}\|>m^\alpha n^\beta\} \text{ by (2.1)}$$

$$=C\frac{1}{m^\alpha n^\beta}\sum_{i,j=1}^{m,n}i^{\alpha-1}j^{\beta-1}m^\alpha m^\beta i^{1-\alpha}j^{1-\beta}P\{i^{\alpha-1}j^{\beta-1}\|X_{ij}\|>m^\alpha n^\beta\}\to 0$$

(by (2.2) and (3.7)).

For (3.2), by Jensen's inequality for conditional expectation, we get

$$\frac{1}{(A_m^\alpha A_n^\beta)^p}\sum_{i=1}^{m}\sum_{j=1}^{n}E\|Y_{ij}^{mn}-E(Y_{ij}^{mn}\mid \mathcal{G}_{ij})\|^p\le\frac{C}{(A_m^\alpha A_n^\beta)^p}\sum_{i=1}^{m}\sum_{j=1}^{n}E\|Y_{ij}^{mn}\|^p$$

$$\le \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m}\sum_{j=1}^{n} E(i^{p(\alpha-1)} j^{p(\beta-1)} \| X_{ij} \|^{p}\ I(i^{\alpha-1} j^{\beta-1} \| X_{ij} \| \le m^{\alpha} n^{\beta}))$$

$$= \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m}\sum_{j=1}^{n} i^{p(\alpha-1)} j^{p(\beta-1)} \sum_{k=1}^{mn^{\beta/\alpha}} \| X_{ij} \|^{p} I((k-1)^{\alpha} < i^{\alpha-1} j^{\beta-1} \| X_{ij} \| \le k^{\alpha})$$

$$\le \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m}\sum_{j=1}^{n} \sum_{k=1}^{mn^{\beta/\alpha}} k^{p\alpha} P((k-1)^{\alpha} < i^{\alpha-1} j^{\beta-1} \| X \| \le k^{\alpha})$$

$$= \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m}\sum_{j=1}^{n} \sum_{k=1}^{mn^{\beta/\alpha}} k^{p\alpha} (P(i^{\alpha-1} j^{\beta-1} \| X \| > (k-1)^{\alpha}) - P(i^{\alpha-1} j^{\beta-1} \| X \| > k^{\alpha}))$$

$$\le \frac{C}{m^{\alpha p} n^{\beta p}} \sum_{i=1}^{m}\sum_{j=1}^{n} \sum_{k=1}^{mn^{\beta/\alpha}} k^{p\alpha-1} P(i^{\alpha-1} j^{\beta-1} \| X \| \ge k^{\alpha})$$

$$\le \frac{C}{m^{\alpha} n^{\beta}} \sum_{i=1}^{m}\sum_{j=1}^{n} i^{\alpha-1} j^{\beta-1} \frac{1}{m^{p\alpha-\alpha} n^{p\alpha-\alpha}} \sum_{k=1}^{mn^{\beta/\alpha}} k^{p\alpha-\alpha-1} (k^{\alpha} i^{1-\alpha} j^{1-\beta} P(\| X \| \ge k^{\alpha} i^{1-\alpha} j^{1-\beta}))$$

$\to 0$ as $m,n \to \infty$.

By applying Theorem 3, the proof is completed. □

The following result is a random index version of Theorem 7.

Theorem 8. *Let* $1 \le p \le 2$, $0 < \alpha, \beta \le 1$, $\mathbb{E}$ *be a p-uniformly smooth Banach space. Suppose that* $\{X_{mn}; m \ge 1, n \ge 1\}$ *is stochastically dominated by a random element X. Suppose that* $\{T_n; n \ge 1\}$ *and* $\{\tau_n; n \ge 1\}$ *are sequences of positive integer-valued random variables such that*

$$\lim_{n\to\infty} P\{T_n > n\} = \lim_{n\to\infty} P\{\tau_n > n\} = 0. \tag{3.6}$$

If

$$\lim_{n\to\infty} nP \| X \| > n = 0,$$

then

$$\max_{\substack{1\le k\le T_m \\ 1\le l\le \tau_n}} \frac{1}{A_m^{\alpha} A_n^{\beta}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l} A_{m-i}^{\alpha-1} A_{n-j}^{\beta-1} (X_{ij} - E(Y_{ij}^{mn} \mid \mathcal{G}_{ij})) \right\| \xrightarrow{P} 0 \text{ as } m \vee n \to \infty.$$

Proof. By the same argument in the proof of Theorem 7 and using Theorem 6. □

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